# Covering Points with Disjoint Unit Discs

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# The original puzzle (Inaba, 2008)

## １０個の点

あなたは同じ大きさの円形のコインをたくさん持っています。
このとき、平面からどのように１０個の点を選んだとしても
それら全てをカバーするようにコインが置けることを示してください。
但し、コインを立てて置いたり、コイン同士を重ねてはいけません。

# The original puzzle - 10 points (Inaba, 2008)

You have a lot of circular coins of the same size.

I choose any 10 points from the plane.

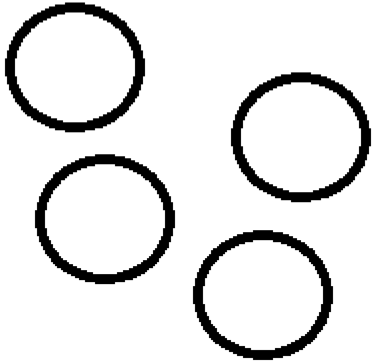
Show that the coins can be placed to cover all of them.

Coins must not overlap.

Finding a covering for a set of points is NP-Complete.
But there is another way.

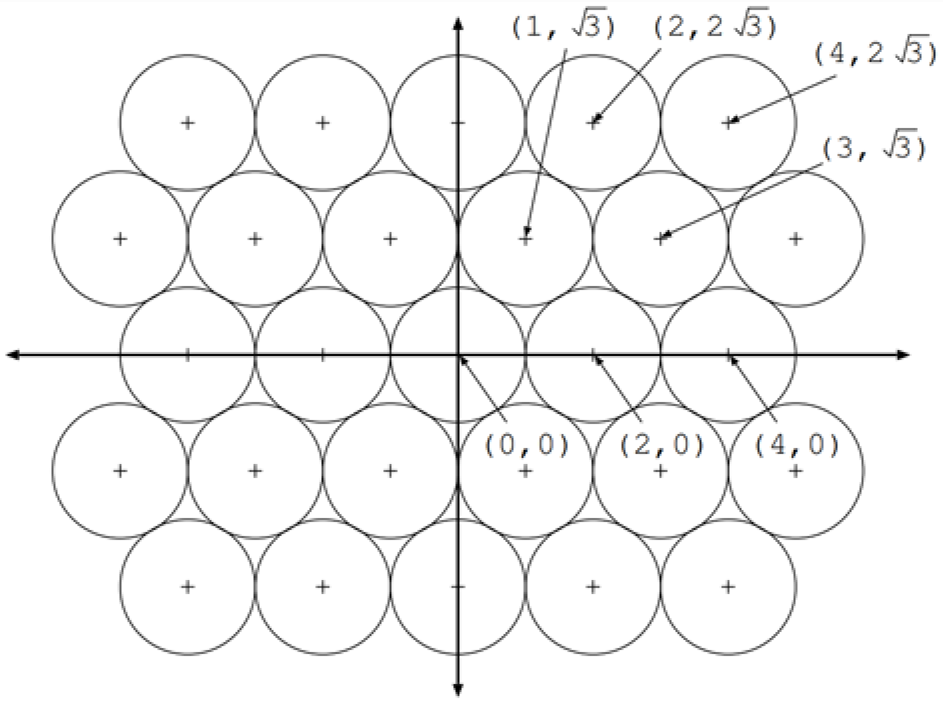
# Proof that 10 points can always be covered

- The proof uses the "probabilistic method", which I'll describe now.
- Suppose I have a random method for placing coins on the plane. If the set of 10 points was impossible to cover, then the probability that the random coins cover it is 0.
- Equivalently, if the random method has a greater than 0 chance of covering the coins, it is possible to cover them!



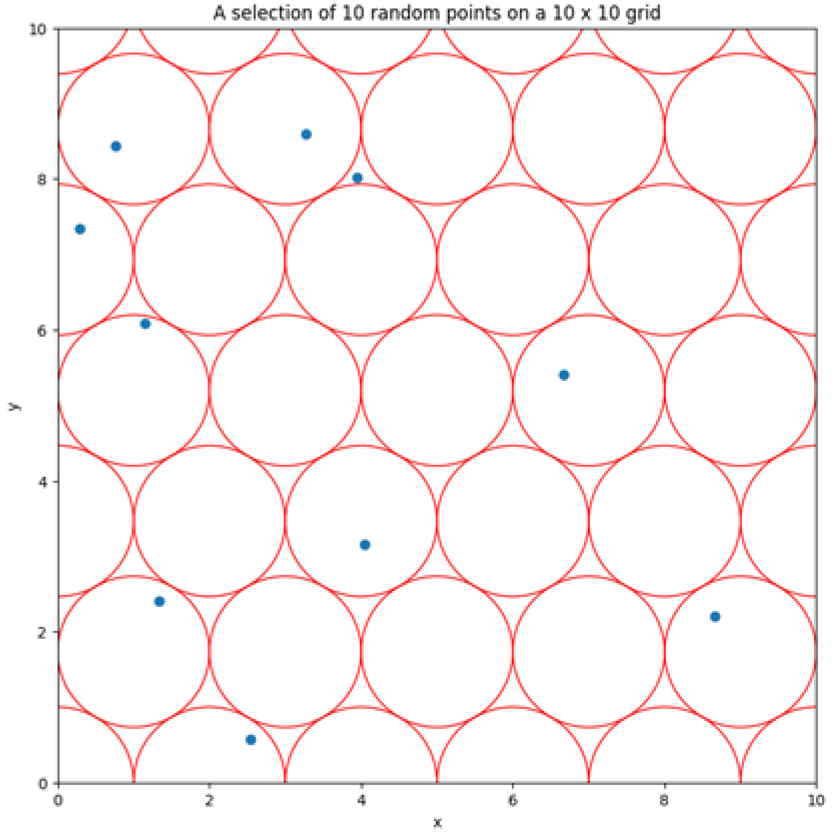
# Proof that 10 points can always be covered

- Start with this infinite hexagonal lattice of coins.
- The coins are “disjoint unit discs”.



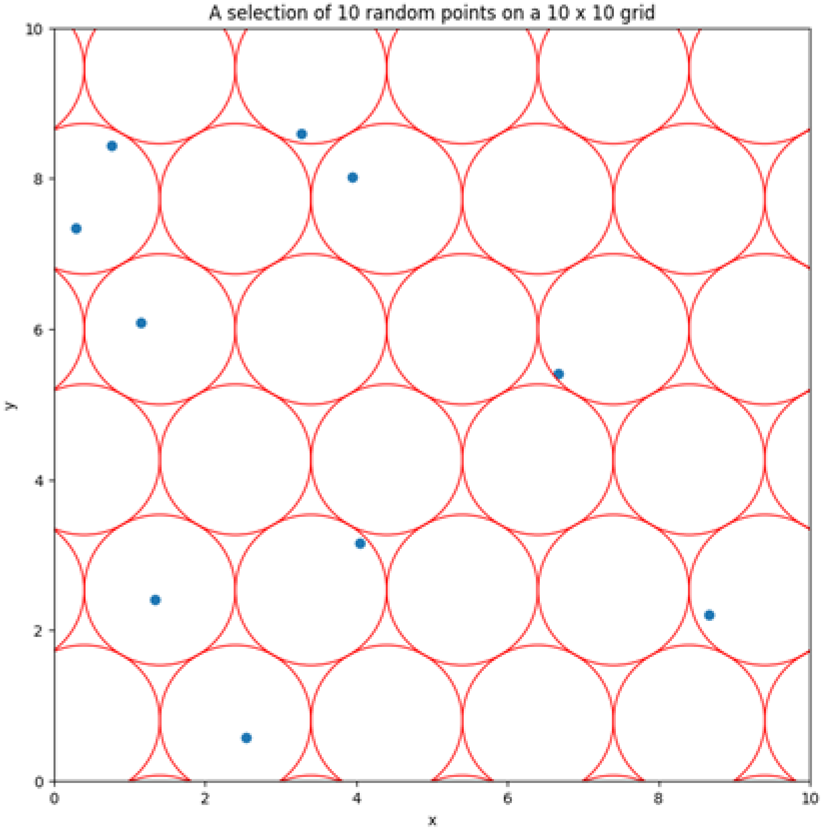
# Proof that 10 points can always be covered

- Here are 10 points as an example.



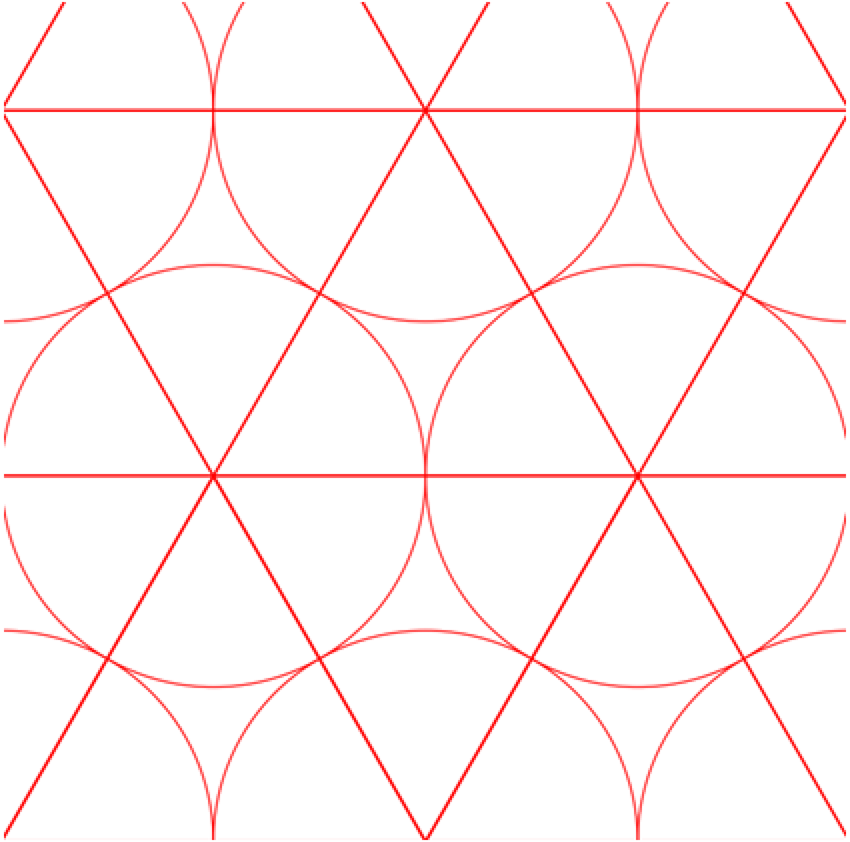
# Proof that 10 points can always be covered

- To make the choice random, translate the lattice so that any lattice is equally likely.
- In this case, all 10 points are now covered.
- We just keep the coins that cover a point - we will need at most 10.



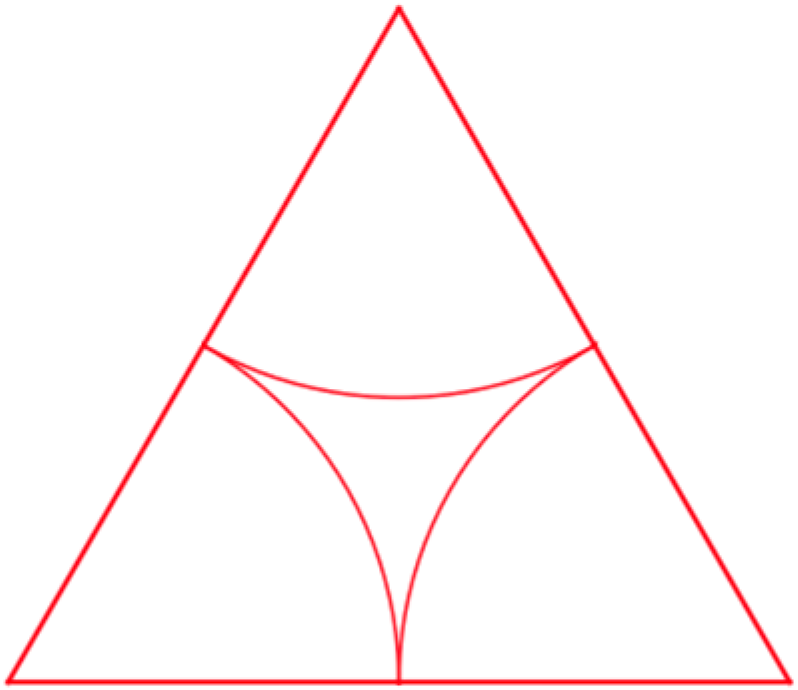
# Proof that 10 points can always be covered

- Each point will be covered by one of these triangles.
- Any lattice covering is equally likely.
- So a given point will land uniformly in one of the triangles. It's covered if it lands in one of the discs.



# Proof that 10 points can always be covered

- The area of the triangle is $\sqrt{3}$.
- The area of each circular sector is $\pi/6$.
- There are 3 sectors, with total area $\pi/2$.
- The probability that a point is covered is $(\pi/2)/\sqrt{3} = \pi\sqrt{3} / 6 = 0.9068997\ldots$



# Proof that 10 points can always be covered

$$\mathbb{P}(A_1) = \frac{\pi\sqrt{3}}{6} > 0.906$$

$$\mathbb{P}(A_1^c) < 0.094$$

$$\mathbb{P}(A_1 \wedge A_2 \wedge \cdots A_{10}) = 1 - \mathbb{P}(A_1^c \vee \cdots \vee A_{10}^c)$$

$$\begin{aligned} \mathbb{P}(A_1^c \vee \cdots \vee A_{10}^c) &\leq \mathbb{P}(A_1^c) + \cdots + \mathbb{P}(A_{10}^c) \\ &\leq 0.94 \end{aligned}$$

$$\mathbb{P}(A_1 \wedge A_2 \wedge \cdots A_{10}) \geq 0.06$$

# 12 points can always be covered

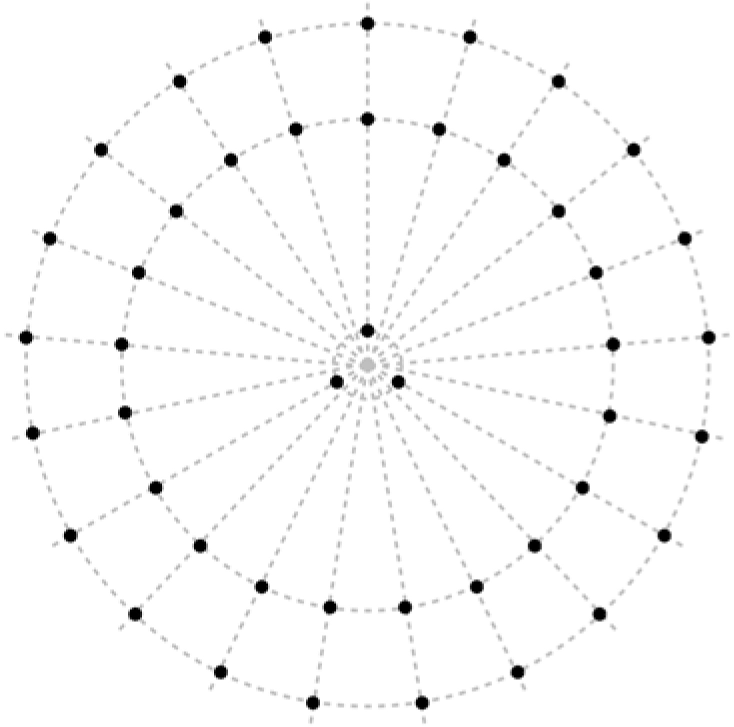
- (Aloupis, Hearn, Iwasawa & Uehara, 2012) showed that any 12 points can be covered.
- This is the best known lower bound for the minimum number of points that can always be covered.

# Some sets of points cannot be covered

- There are some sets of points that cannot be covered.
- As a simple example, make a dense set of points. Any set of coins must leave some of them uncovered.

# There is a set of 45 points that can’t be covered

- (Aloupis, Hearn, Iwasawa & Uehara, 2012) presented a set of 45 points that can’t be covered.
- The outer circle has radius 1.0001.
- This is the smallest known set of points that cannot be covered.



# Conclusion

- There's a short probabilistic proof that any set of 10 points can be covered by disjoint unit discs.
- The smallest number of points that can't be covered by unit discs is at least 13 and at most 45.