A generalization of Euler's Pentagonal number theorem

$$\prod_{n=1}^{\infty}(1-q^n) = 1+\sum_{k=1}^{\infty}(-1)^k\left(q^{f(k)}+q^{f(-k)}\right)$$

$$= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} \cdots$$

where $f(k) = \frac{3k^2-k}{2}$

$\cdots$	$f(-2)$	$f(-1)$	$f(0)$	$f(1)$	$f(2)$	$\cdots$
$\cdots$	7	2	∅	1	5	$\cdots$

$F(k) \equiv \{2, ∅, 1\}$

Euler's PNT

Relation to partitions

$n =$	0	1	2	3	4	5	$\cdots$
$p(n) =$	1,	1,	2,	3,	5,	7,	$\cdots$
	$\{\}$	(1)	(2)	(3)	(4)	(5)	
			(1,1)	(2,1)	(3,1)	(4,1)	
				(1,1,1)	(2,2)	(3,2)	
					(2,1,1)	(3,1,1)	
					(1,1,1,1)	(2,2,1)	
						(2,1,1,1)	
						(1,1,1,1,1)	

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5$$

$$= \sum_{k=0}^{\infty} p(k) q^k$$

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Generalizing from $f(k)=\frac{3k^2-k}{2}$

to

$$f(\alpha,\beta,k)=\frac{(\alpha+\beta)k^2-(\beta-\alpha)k}{2}$$

where $\alpha<\beta\in\mathbb{N}$

induces a structure through the theory of partitions to give us natural definitions for

$p(\alpha,\beta,n)$ and $\sigma(\alpha,\beta,n)$

as partitions/divisors such that

$$\lambda\equiv\alpha,\beta \text{ (OR) } 0 \mod (\alpha+\beta)$$

$$d\equiv\alpha,\beta \text{ (OR) } 0 \mod (\alpha+\beta)$$

note the minimum pair, $\alpha=1$, $\beta=2$ corresponds to the unrestricted functions because $n\equiv1,$ zero mod 3

Examples

$\alpha = 1$, $\beta = 3$, $f(1,3,k) = \frac{4k^2-2k}{2} = 2k^2 - k$

or value rep $\equiv \{\beta, 0, \alpha\} = \{3, 0, 1\}$

$k =$	…	-3	-2	-1	0	1	2	3	4	…
	…	21	10	$\{3$	0	$1\}$	6	15	28	…

hexagonal #'s

generalized hexagonal #'s

aka the $\binom{n}{2}$ binomials for $n > 0$

aka the triangular #s

note: Fixing $\alpha = 1$ isolates the quadratics of the g-gonal numbers where $g = \beta + 3$, so

	α, β
pentag	1, 2
hexag	1, 3
septag	1, 4
⋮	⋮

$$f_{\alpha\beta}(k) \equiv \{3, \emptyset, 1\}$$

$$S_{\alpha\beta} \equiv n \in \mathbb{N} \mid n \equiv \alpha, \beta, \emptyset \pmod{(\alpha+\beta)}$$

$$n \in \mathbb{N} \quad n \equiv 1, 3, \emptyset \pmod{4}$$

$$\prod_{n=1}^{\infty} \left(1 - q^{(\alpha+\beta)n-\beta}\right)\left(1 - q^{(\alpha+\beta)n-\alpha}\right)\left(1 - q^{(\alpha+\beta)n}\right)$$

$$= 1 + \sum_{k=1}^{\infty} (-1)^k \left(q^{2k^2-k} + q^{2k^2+k}\right)$$

$$= 1 - q - q^3 + q^6 + q^{10} - q^{15} - q^{21} \cdots$$

1 −1 0 −1 ∅ ∅ 1 ∅ ∅ ∅ 1 ∅ ∅ ∅ ∅ −1 ∅ - - -

1	0	0	0	0	0	0	0	0	0	0
1	-1	0	-1	0	0	1	0	0	0	1
	1	-1	0	-1	0	0	1	0	0	0
		1	-1	0	-1	0	0	1	0	0
			2	-2	0	-2	0	0	2	0
				3	-3	0	-3	0	0	3
					4	-4	0	-4	0	0
						5	-5	0	-5	0
							7	-7	0	-7
								10	-10	0
									13	-13
										16

$$\prod_{n \in S_{1,8}} \frac{1}{1-q^n} = \sum_{k=0}^{\infty} p(1,3,k)\, q^k =$$

$$\{1, 1, 1, 2, 3, 4, 5, 7, 10, 13, 16, \cdots\}$$

k =	0	1	2	3	4	5	6	7	8	9	10
P(1,3,k)	1	1	1	2	3	4	5	7	10	13	16
		1	1	1	2	3	4	5	7	10	13
		1	1	3	3	3	6	9	12	15	21
T(1,3,k)							-6	-6	-6	-12	-18
				4	5						-10
						6	4	8	13	13	
											6

1	1	1	1	1	1	1	1	1	1
	2	3	2	5	2	7	2	3	2
			4		3		4	9	5
					6		8		10

$$\bar{p}(n) = p(1,1,n)$$

generalizes over same α, β

$$\bar{p}(\alpha,\beta,n) = \prod_{n \in S_{\alpha,\beta}} \frac{1+q^n}{1-q^n}$$

where $\bar{p}(n) = \bar{p}(1,2,n)$

So what is $\bar{p}(1,1,n)$?

$$\bar{\bar{p}}(n) = \bar{p}(1,1,n) = \prod_{n=1}^{\infty} \frac{(1+q^k)^4}{1-q^{4k}}$$

$\bar{p}(n)$	1	2	4	8	14	24	40	64	154
$\bar{\bar{p}}(n)$	1	4	10	24	52	104	200	368	654

What is $p(1,1,n)$?

$16, 9, 4 \{1, 0, 1\} 4, 9, 16$

4-gonal – Square #'s

$$\sum_{n=0}^{\infty} p(1,1,n) q^n = \prod_{k=1}^{\infty} \frac{1-q^{2k}}{(1-q^{2k-1})^2} = \prod_{k=1}^{\infty} \frac{1+q^k}{1-q^k}$$

$p(1,1,n) = \bar{p}(n)$, # overpartitions of n

$n =$	0	1	2	3	4
$p(n) =$	1	1	2	3	5
	$\{\}$	1	2	3	4
			11	21	31
				111	22
					211
					1111

$\overline{p}(n)$	1	2	4	8	14
	$(\{\})$	(1)	(2)	(3)	$(4, \overline{4})(4, 4)$
		$(\overline{1})$	$(\overline{2})$	$(\overline{3})$	$(31), (\overline{3}1), (3\overline{1}), (\overline{3}, \overline{1})$
			(11)	(21)	$(22), (\overline{2}2)$
			$(\overline{1}1)$	$(\overline{2}1)$	$(211), (\overline{2}11), (2\overline{1}1), (\overline{2}\overline{1}1)$
				$(2\overline{1})$	$(1111), (\overline{1}111)$
				$(\overline{2}\overline{1})$	
				(111)	
				$(\overline{1}11)$	

Overpartitions

1   2   4   8

{}   (1)   (2), $(\overline{2})$ (3), $(\overline{3})$

$(\overline{1})$ (1 1), $(\overline{1}\ 1)$ (2 1) $(\overline{2}\ 1)$ $(\overline{2}\ \overline{1})$ $(2\ \overline{1})$

(1 1 1) $(\overline{1}\ 1\ 1)$

2-overpartitions

1   4   10   26

{}

(1) $(\overline{1})$ $(\underline{1})$ $(\underline{\overline{1}})$

(2) $(\overline{2})$ $(\underline{2})$ $(\underline{\overline{2}})$ (1 1) $(\overline{1}\ 1)$ $(\underline{1}\ 1)$ $(\underline{\overline{1}}\ 1)$ $(\underline{\overline{1}}\ \underline{1})$ $(\overline{1}\ \underline{1})$

(3) $(\overline{3})$ $(\underline{3})$ $(\underline{\overline{3}})$

(2 1) $(\underline{2}\ 1)$ $(2\ \underline{1})$ $(\underline{2}\ \underline{1})$

$(\overline{2}\ 1)$ $(\underline{\overline{2}}\ 1)$ $(\overline{2}\ \underline{1})$ $(\underline{\overline{2}}\ \underline{1})$

$(2\ \overline{1})$ $(\underline{2}\ \overline{1})$ $(2\ \underline{\overline{1}})$ $(\underline{2}\ \underline{\overline{1}})$

$(\overline{2}\ \overline{1})$ $(\underline{\overline{2}}\ \overline{1})$ $(\overline{2}\ \underline{\overline{1}})$ $(\underline{\overline{2}}\ \underline{\overline{1}})$

(1 1 1) $(\overline{1}\ 1\ 1)$

$(\underline{1}\ 1\ 1)$ $(\underline{\overline{1}}\ 1\ 1)$ $(\overline{1}\ \underline{1}\ 1)$ $(\underline{\overline{1}}\ \underline{1}\ 1)$

Let $D_{\pi(i)}$ be # distinct parts of partition λ_i of n

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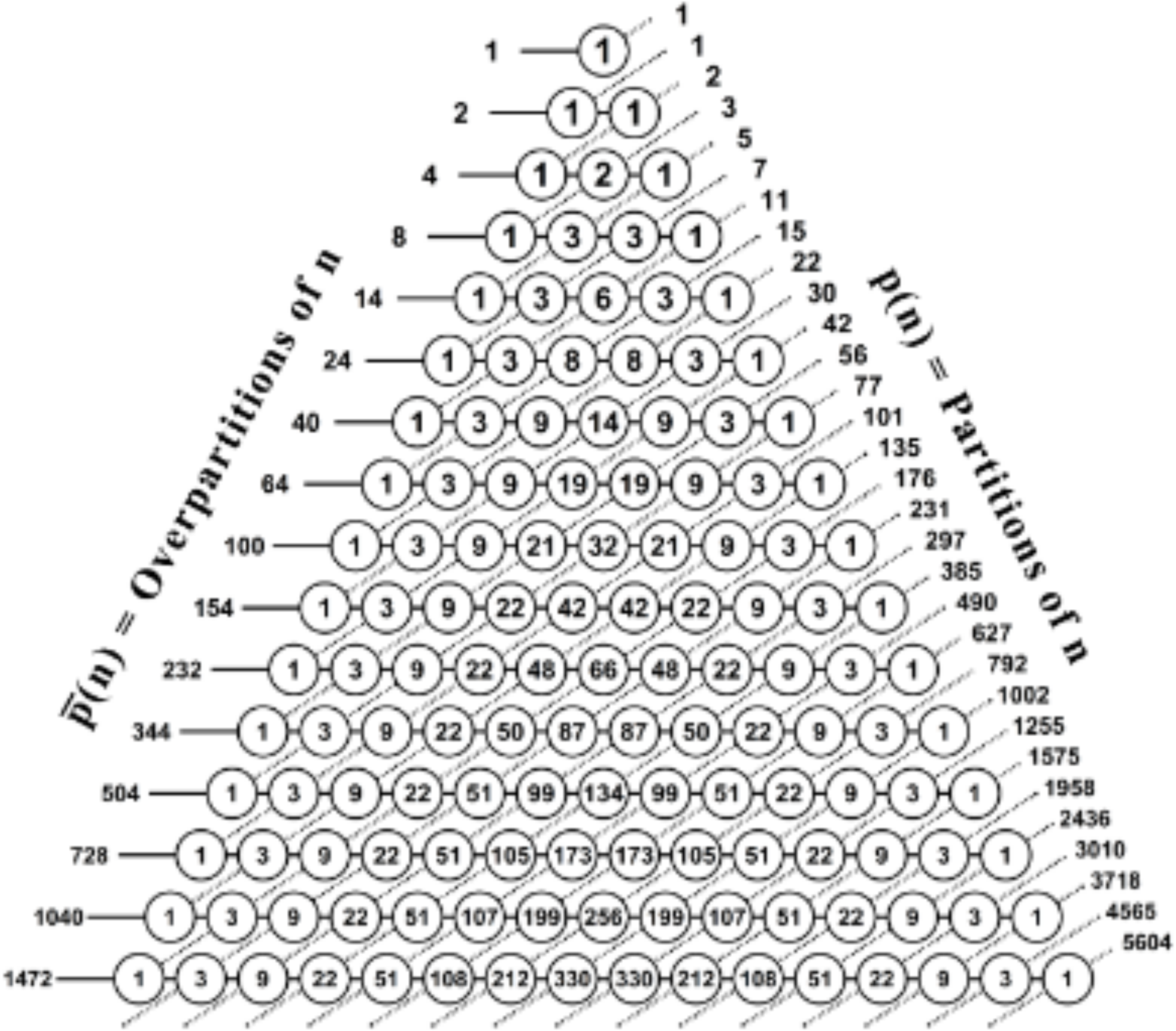
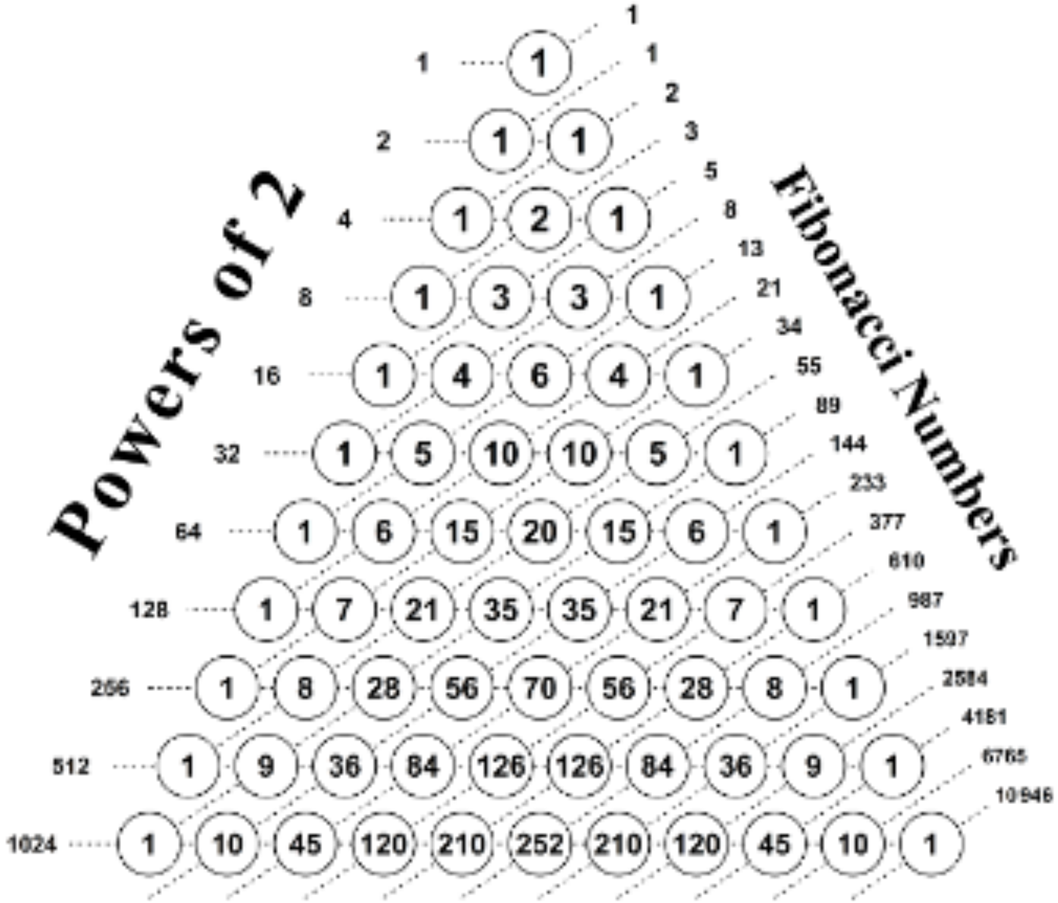
then $\bar{p}(n) = \sum_{i=1}^{p(n)} 2^{(D_{\pi_i})}$

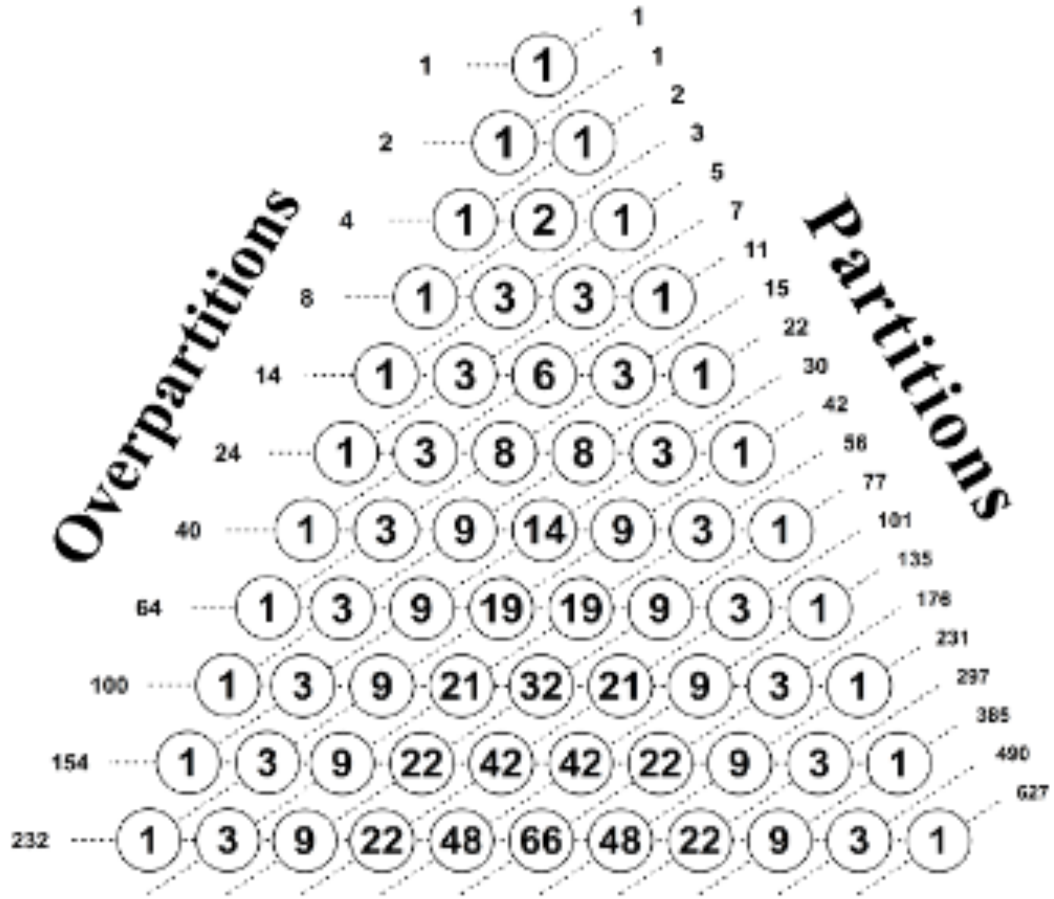
$$\left.\begin{array}{lll} 3 & 1 & 2^1 \\ 2\,1 & 2 & 2^2 \\ 1\,1\,1 & 1 & 2^1 \end{array}\right\} \Sigma = 8 \,,\quad \bar{p}(3) = 8$$

Partnomial Coefficients Triangle



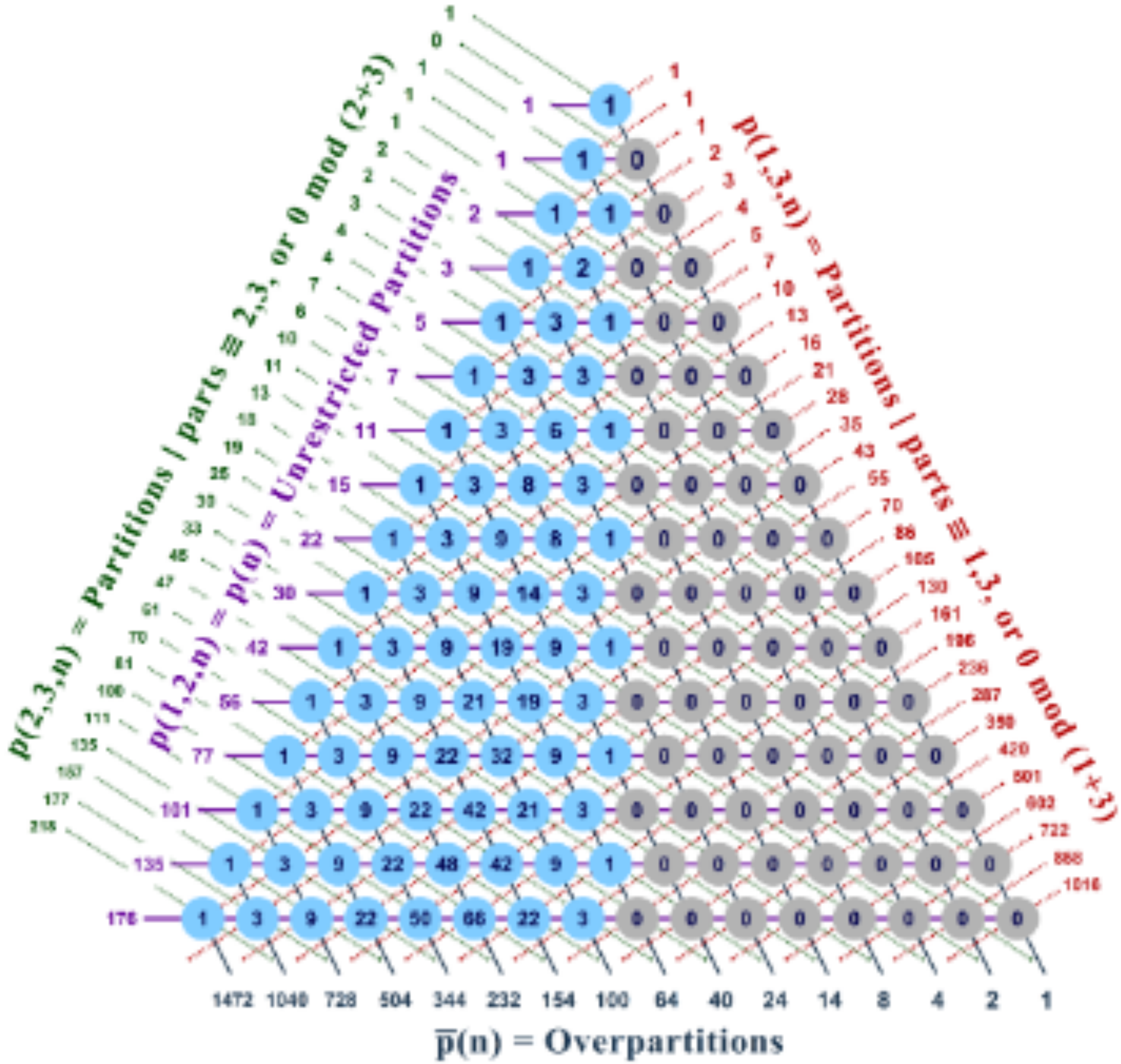


Binomial Coefficients

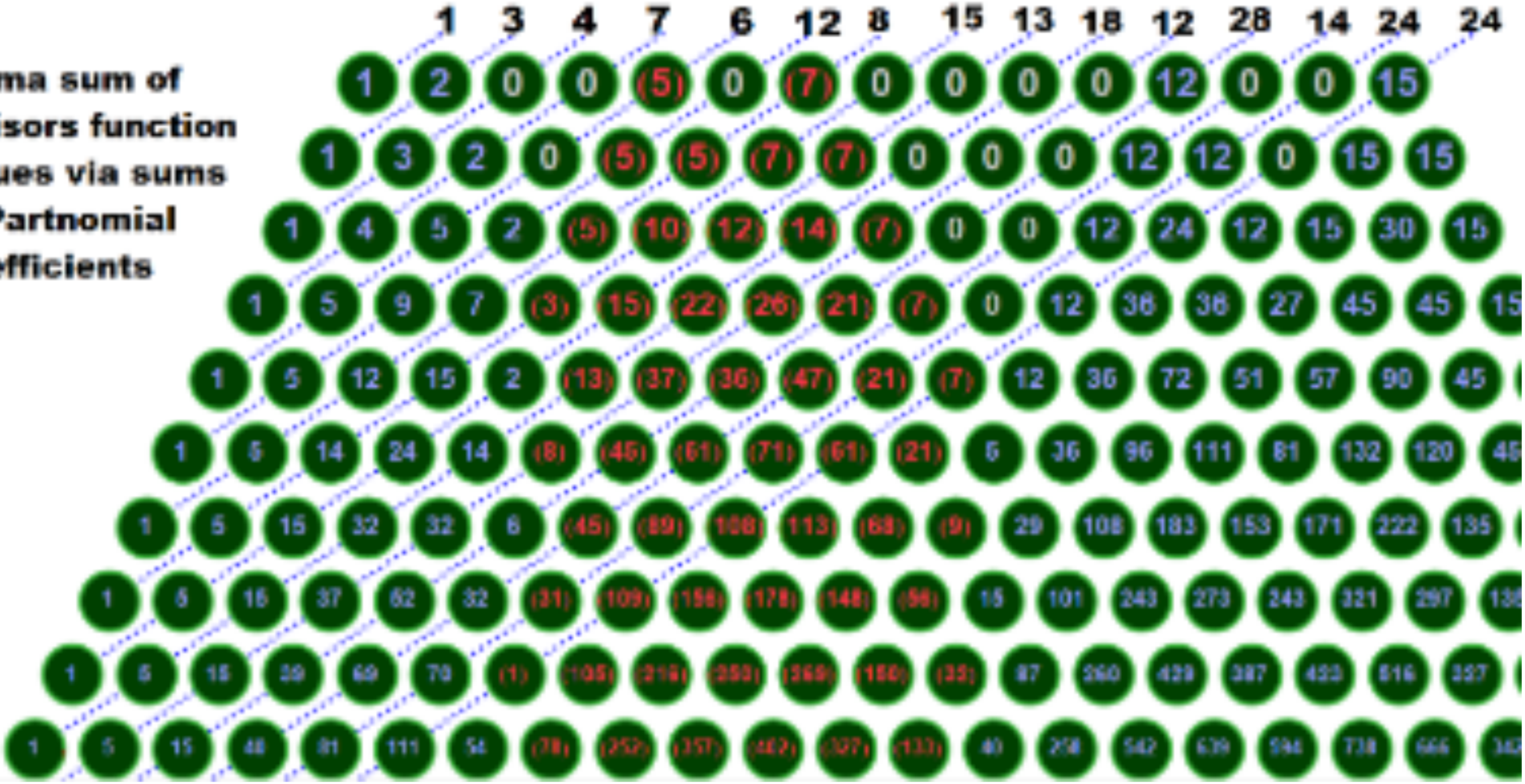


Partnomial Coefficients

Partnomial Coefficients Half Triangle



Sigma sum of divisors function values via sums of Partnomial Coefficients



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warm = Table[N[Sum[DivisorSigma[1, k], {k, x^2, x^2 + 2 x}] / ((x^2 + x + 1) (2 x + 1)), 7],
    {x, 1001, 1999, 2}];

cold = Table[N[Sum[DivisorSigma[1, k], {k, x^2, x^2 + 2 x}] / ((x^2 + x + 1) (2 x + 1)), 7],
    {x, 1002, 2000, 2}];

ListPlot[{warm, cold, Table[Zeta[2], {j, 1, 500}]}, Joined → True]
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