A method for solving quadratic equations

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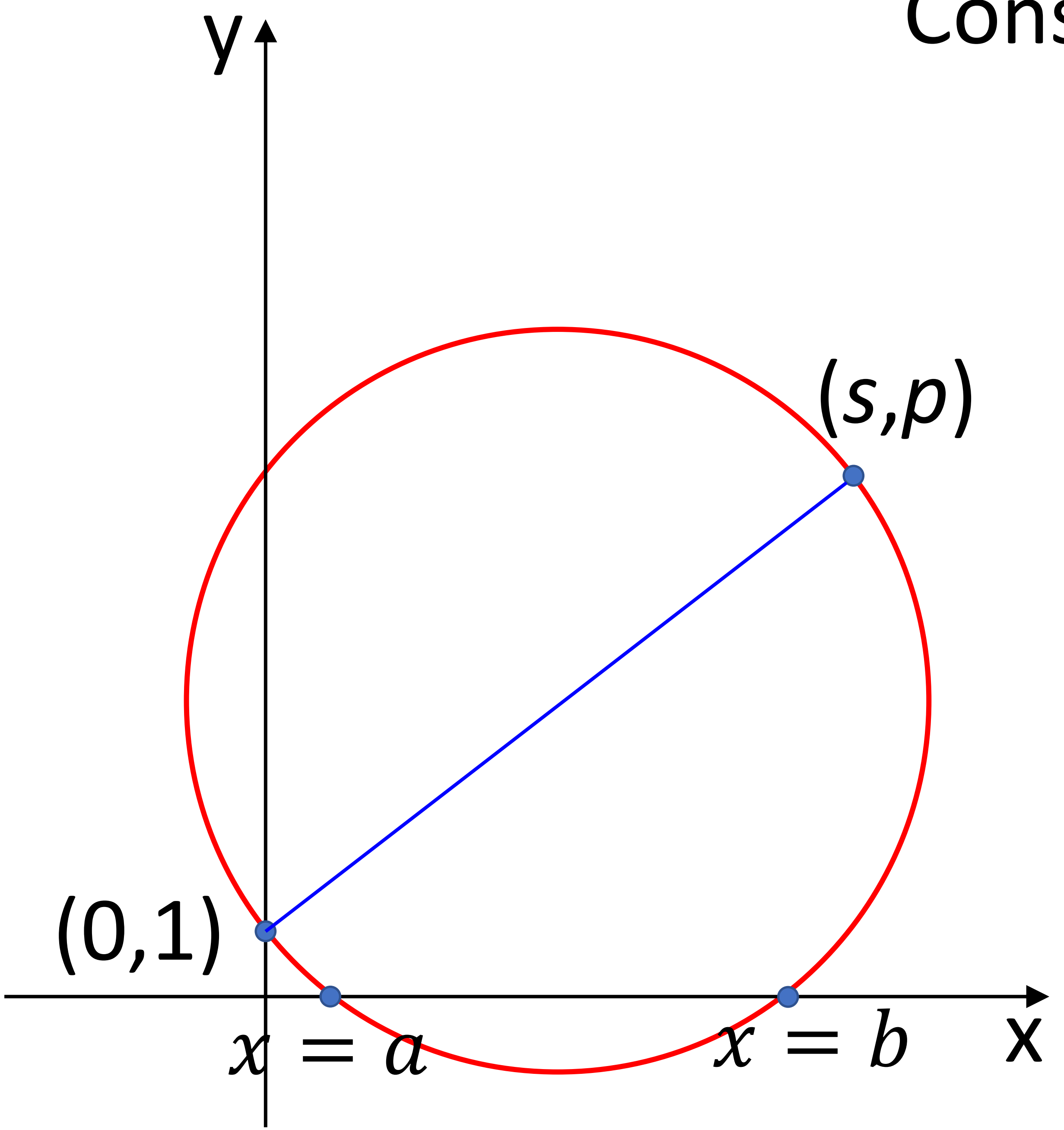


Consider a quadratic equation

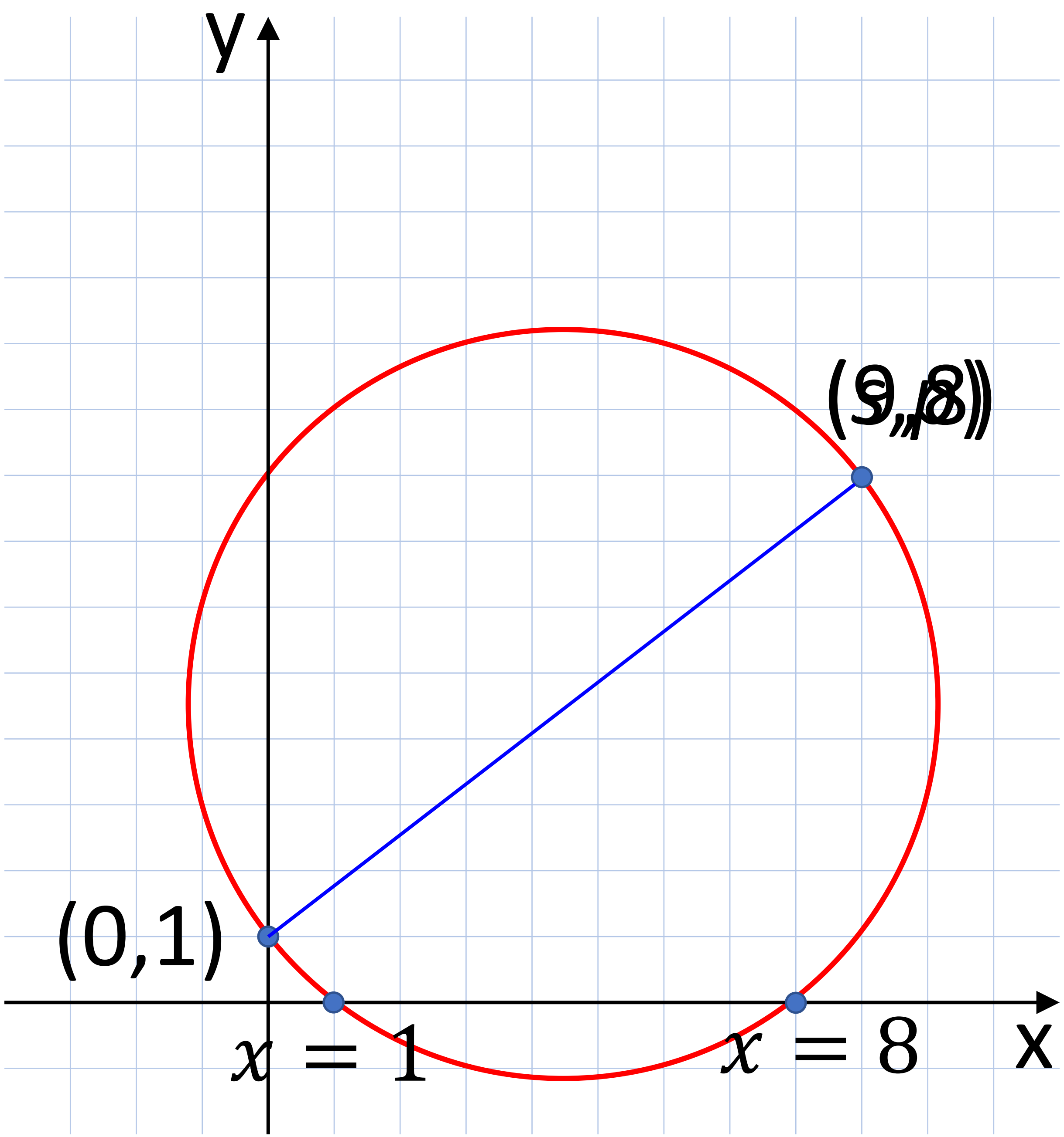
$$x^2 - sx + p = 0$$

with roots *a*, *b*

(i.e. *s* is the sum of the roots and *p* is their product.)

Construct a circle with diameter (0,1), (*s*,*p*).

The *x* intercepts are the roots of the equation.

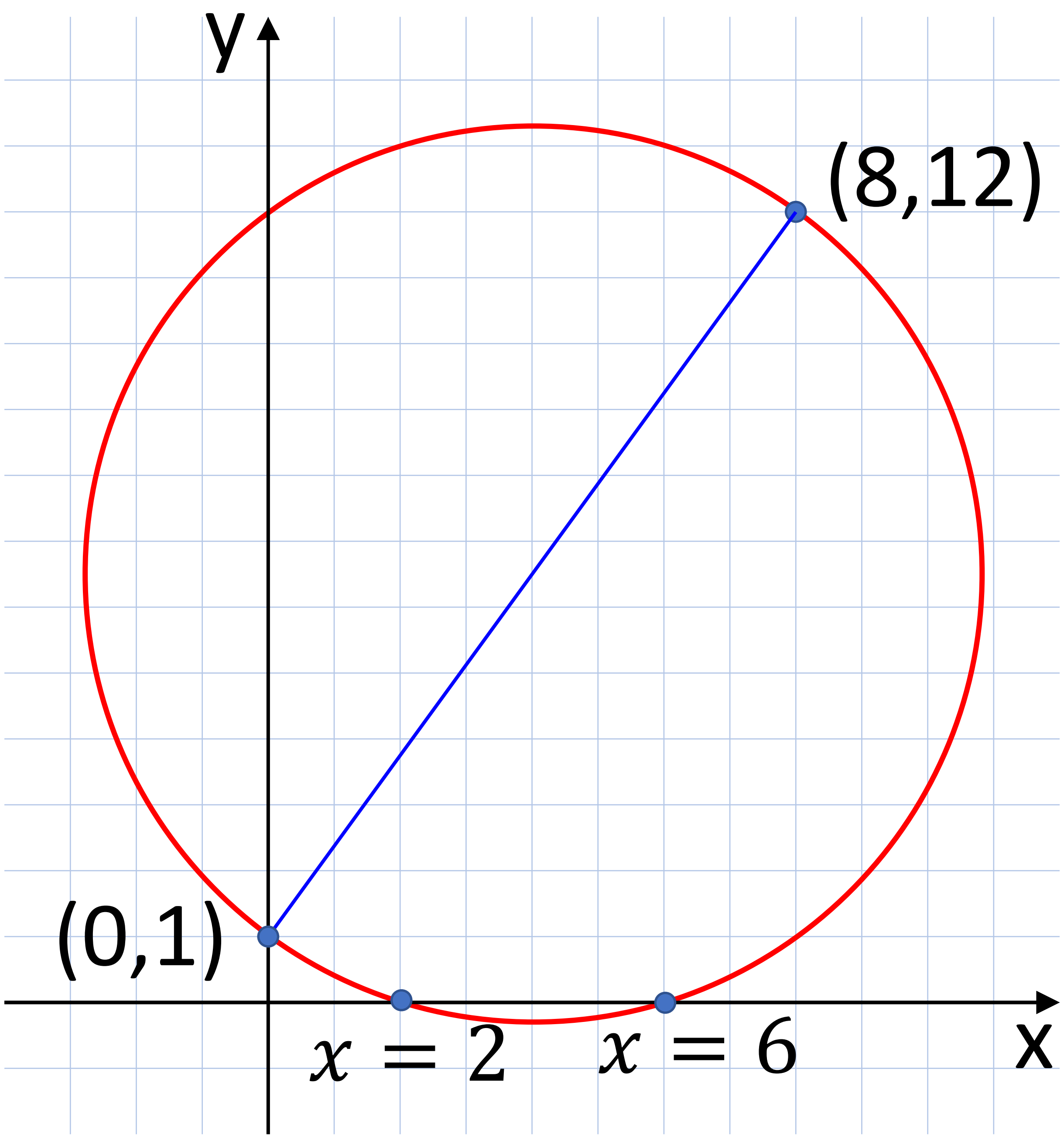


$$x^2 - 9x + 8 = 0$$

$$s = 9$$

$$p = 8$$

Roots are x=1, x=8



Another example

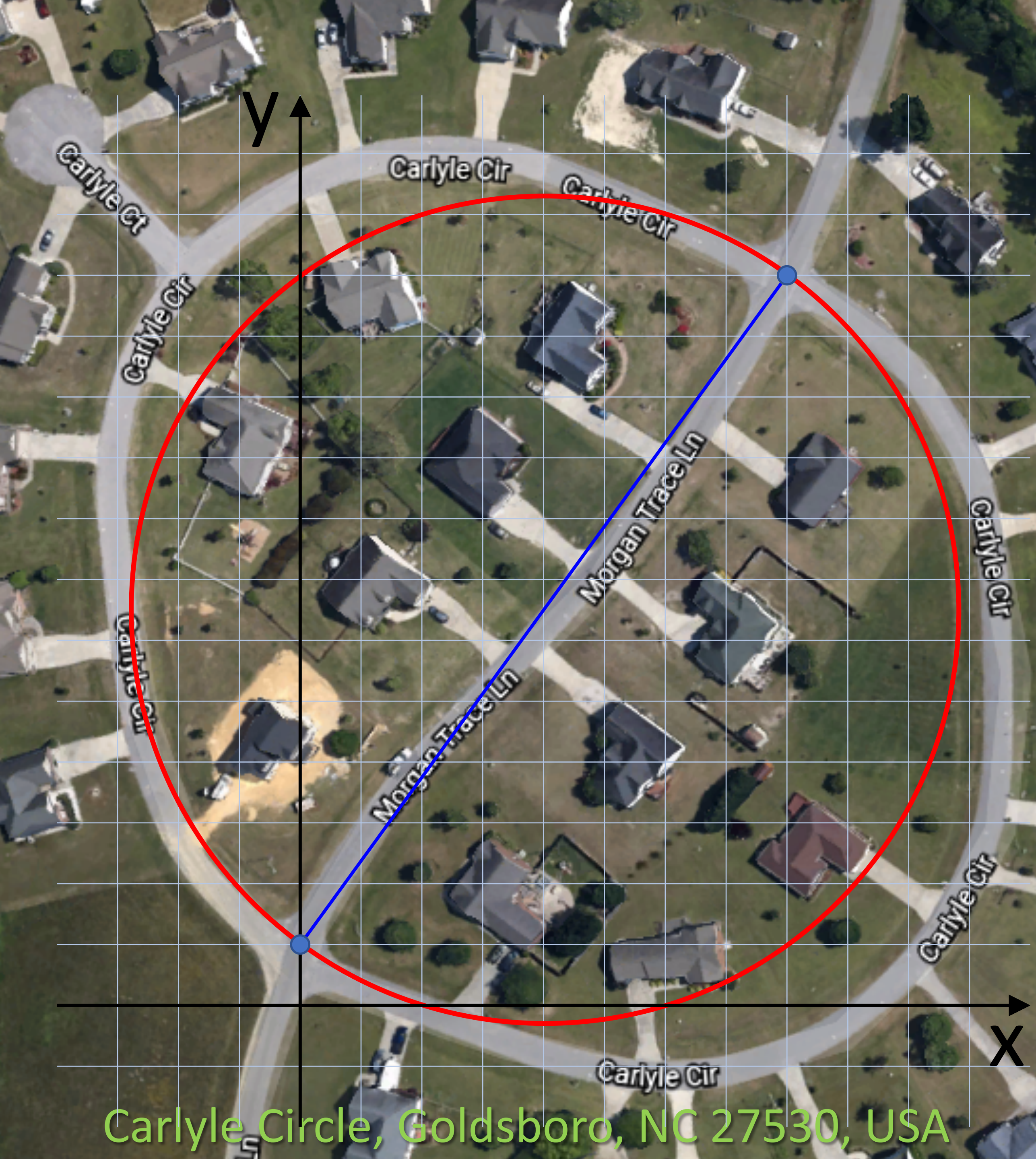
$$x^2 - 8x + 12 = 0$$

$$s = 8$$

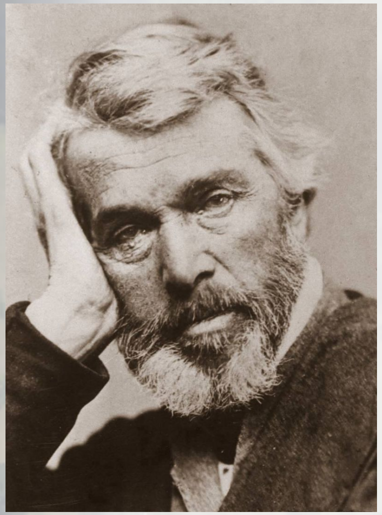
$$p = 12$$

Roots are x=2, x=6

Carlyle Circle



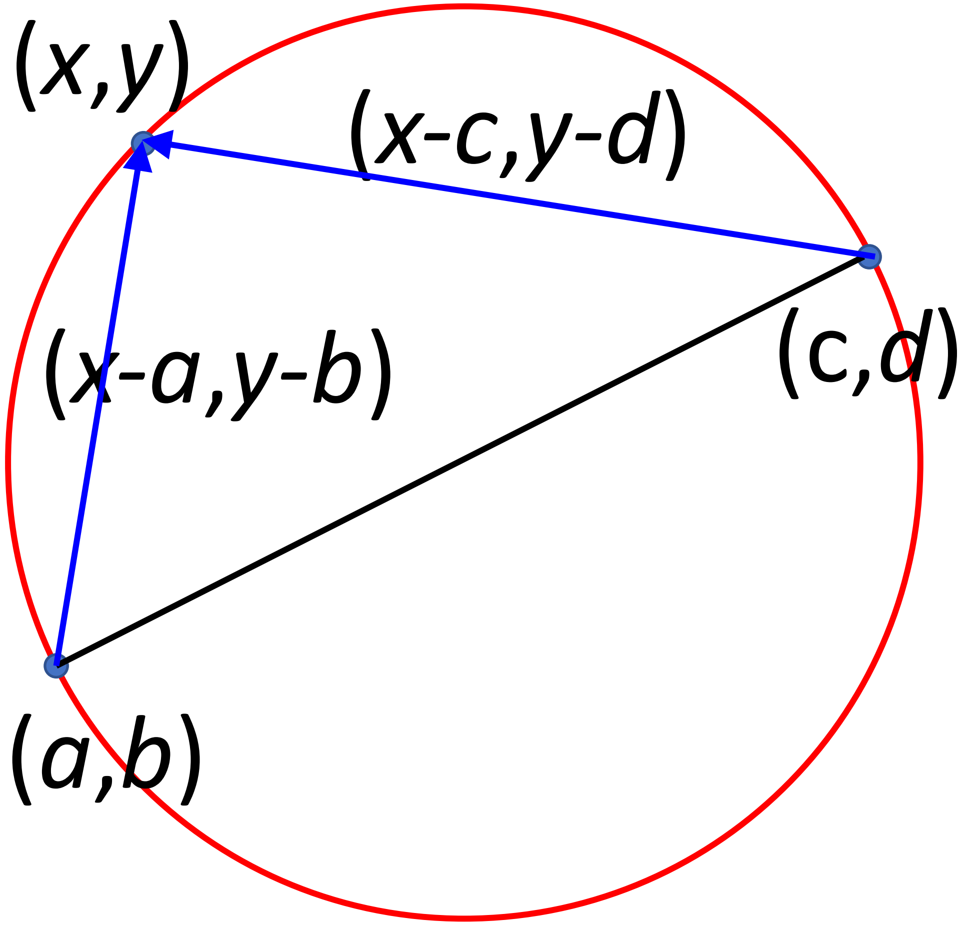
Carlyle Circle, Goldsboro, NC 27530, USA



Thomas Carlyle, 1795-1881

- Scottish historian, writer, mathematician
- His 1837 history of The French Revolution was the inspiration for Dickens' A Tale of Two Cities
- Coined the term "the dismal science" for economics

Why does it work?



Circle with diameter (a,b),(c,d)

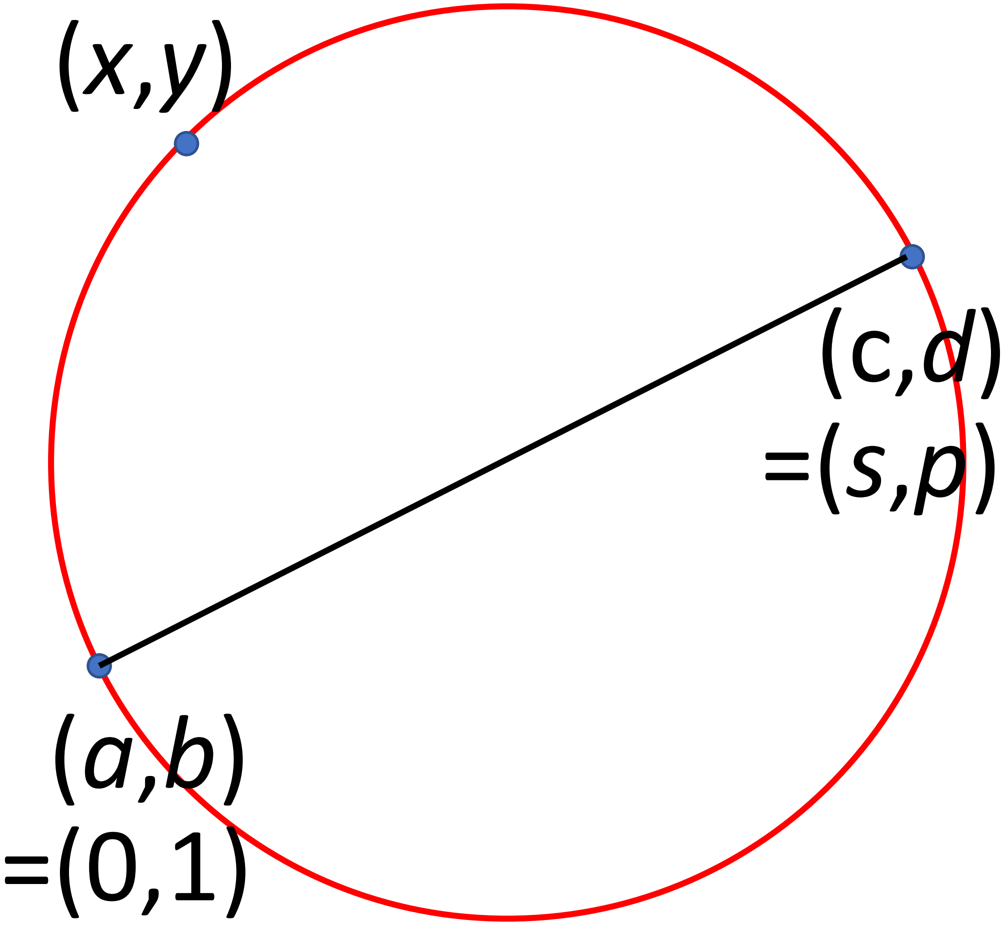
Vectors (x-a, y-b) and (x-c,y-d) meet at a right angle.

So their dot product is 0.

Formula for general point (*x*,*y*):

$$(x-a)(x-c)+(y-b)(y-d)=0$$

Applying it to the Carlyle circle



Let $(a, b) = (0,1)$
$(c, d) = (s, p)$

$\therefore x(x - s) + (y - 1)(y - p) = 0$

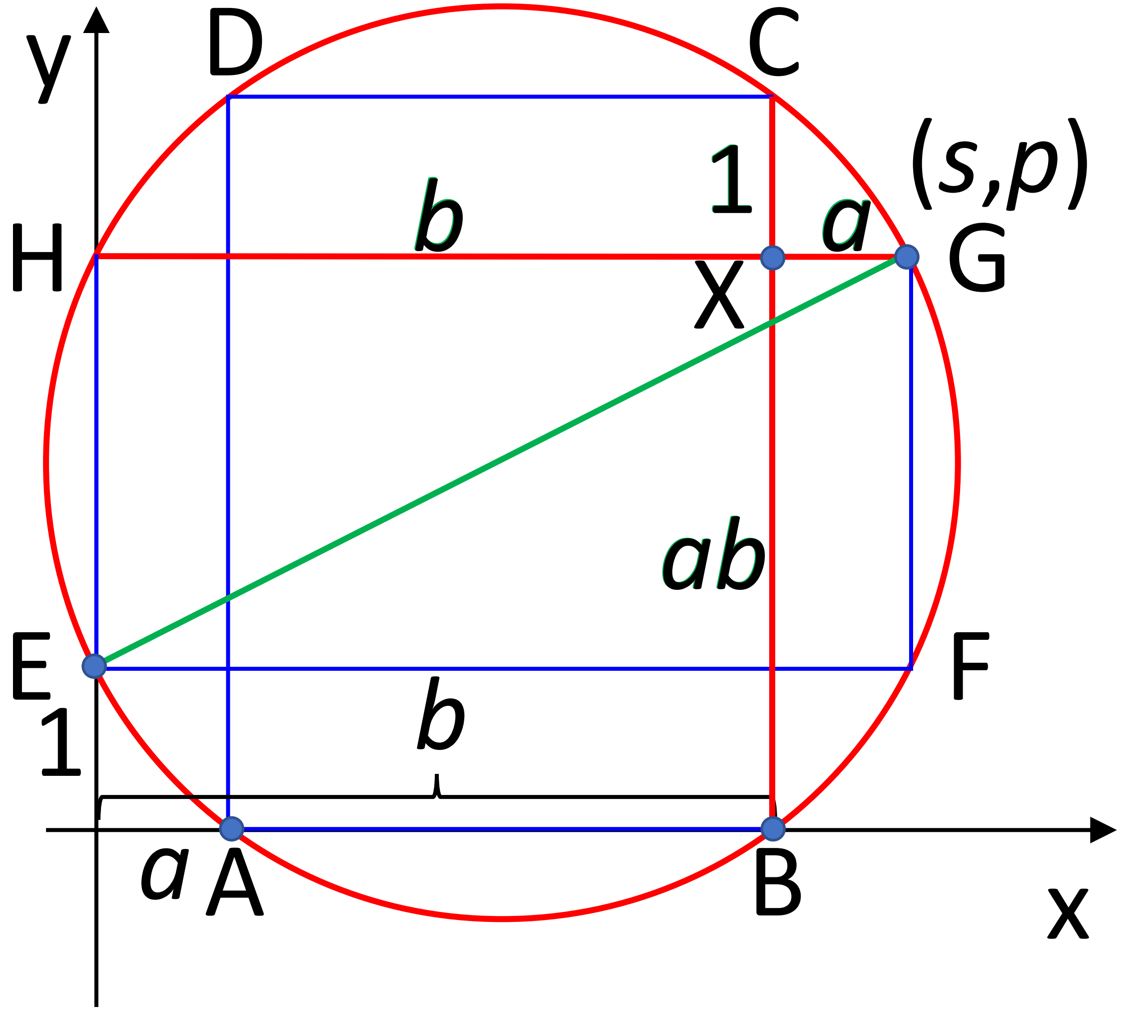
On x axis, $y = 0$

$\therefore x^2 - sx + p = 0$

$(x - a)(x - c) + (y - b)(y - d) = 0$

So the x intercepts are the roots of this equation

A geometrical proof



Given x intercepts A,B with x coordinates *a*,*b*

construct rectangle ABCD.

Given E with coordinates (0,1)

construct rectangle EFGH.

By symmetry, HX = *b*, XG = *a*, *XC*=1.

By the intersecting chord theorem,

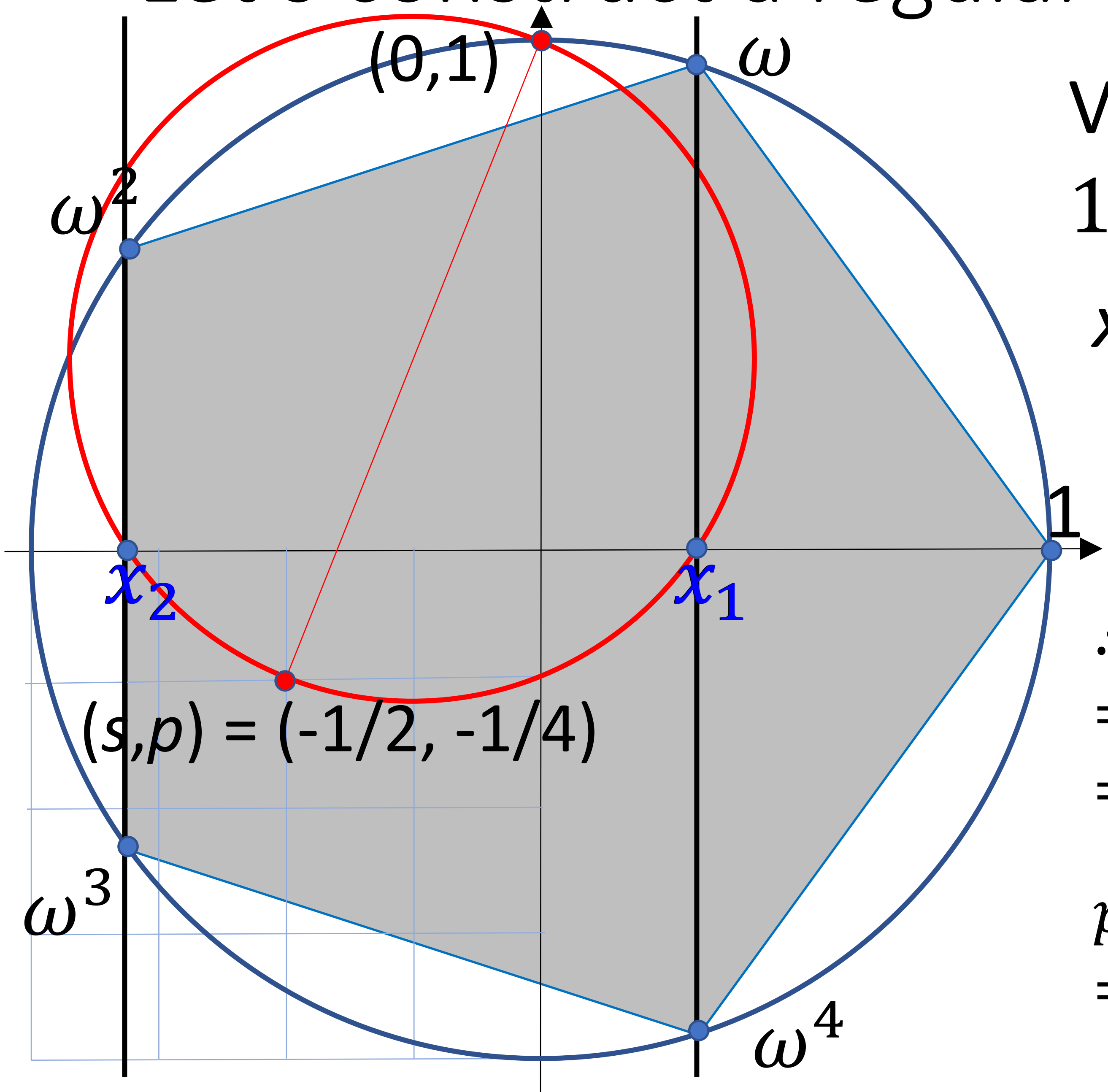
$BX \times 1 = ab$

So G has coordinates

$s = HX+XG = b+a$, $p = BX = ab$

So a circle with diameter EG, coordinates (0,1), (*s*,*p*) has *x* intercepts *a*,*b*.

Let's construct a regular pentagon in a circle



Vertices are 5th roots of unity:

$1, \omega, \omega^2, \omega^3, \omega^4$, where $\omega^5 = 1$

x coordinates x_1, x_2

$$\omega + \omega^4 = 2x_1$$
$$\omega^2 + \omega^3 = 2x_2$$

$\therefore s = x_1 + x_2$
$= (\omega + \omega^2 + \omega^3 + \omega^4)/2$
$= -1/2$

$p = x_1 x_2 = (\omega + \omega^4)(\omega^2 + \omega^3)/4$
$= -1/4$

Construction of the regular heptadecagon (17-gon) is left as an exercise for the viewer.

But maybe the basis for some of the classic constructions will now be clearer.