A minus times a minus is a .... pain?

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We all learn fairly young that ‘a minus times a minus is a plus’. This fact causes a certain amount of pain: it seems like an arbitrary rule, and lots of students/pupils find it hard to lose the negativity. Some times they ask ‘Yes, but why?’

That's just the rule.

or, in a more literary form,

Ours not to reason why,
Ours but to multiply.

Clearly, that isn't very satisfactory.
In fact, I want to say

A minus times a minus is a plus
The reason why, we really should discuss.

There are two separate issues here, though:

1. Convincing them it's right (whatever that means), and
2. Getting them to do it (which is a problem I shall run away from very fast).

Here's one argument:

$$
\begin{aligned}
(-1) \times 3 &= -3 \\
(-1) \times 2 &= -2 \\
(-1) \times 1 &= -1 \\
(-1) \times 0 &= 0 \\
(-1) \times (-1) &= \text{what comes next?}
\end{aligned}
$$

(This is in Ed Southall's 'Yes, but Why?') And it is pretty persuasive: the pattern is all but irresistible. But there's a lot of stuff buried in there, and that's what I want to start to explore now.

The first question that springs to my mind is “But why is $(-1) \times n$ the same as $-n$?”

And at that point it strikes me to wonder “Come to think of it, what do I mean by -1, or $-n$?”

Well, I think I know what I mean by n when n is a natural number. So in that case, I guess that $-n$ is some kind of imaginary? no (that doesn’t sound right) …crazy? no (that sounds too informal) …irrational? (well, that doesn’t sound too informal, but I might want to keep it for later) …negative? yes, I like that. It’s a negative number.

And it’s the number I add to n to get 0.

Yes, I definitely like that. $-n$ is, by definition, the thing that makes

$$(-n) + n = 0$$

work: $-n$ is the *additive inverse* of n.

So, I've added in a whole bunch of numbers to my natural numbers.
For each n, I add in $-n$.
I don't need to do it for 0, because $0 + 0 = 0$, so that tells me that $-0 = 0$.
But how do these things behave with the natural numbers I started with? I have to decide what happens when I do arithmetic that combines them in various ways.
Well, they're my numbers, so I guess it's up to me. I can make them do whatever I want. Ah, the feeling of power.

Oh, but wait a minute.
I can't make them do whatever I want, really, at least, not if I'm sensible. After all, the argument up above depended on a certain pattern. I guess I really need the patterns I'm used to with the natural numbers to keep on working. So I want things like associativity and commutativity of addition and multiplication to keep working, and I want the distributive law to hold, just like before.
That will narrow things down a bit.

In fact, it narrows them down completely.
I can now see why $(-1) \times n = -n$.

$$\begin{aligned}(-1) \times n + n &= (-1) \times n + 1 \times n \\ &= ((-1) + 1) \times n \\ &= 0 \times n \\ &= 0\end{aligned}$$

Which is nice.

At least, it is if I'm completely sure about $0 \times n = 0$ no matter what n is. What could the problem be? Well, 0 I could claim that 0 is defined that way. But then how can I be sure that adding 0 doesn't change anything? Or I could say that 0 is defined as the number which makes no difference when I add it. But then how do I know that multiplying by it always gives 0?

It's OK: you can deduce $0 \times n = 0$ for arbitrary n from $0 + n = n$ for arbitrary n.

$$0 \times n = (0 + 0) \times n = 0 \times n + 0 \times n$$

so adding $0 \times n$ didn't make any difference, so it must be 0.

The important thing to note is that they aren't independent: once we've chosen one, we're stuck with the other. Hold that thought.

So now I can see that if n is any number, positive or negative, $(-1) \times n$ is the additive inverse of n, which I can represent as $-n$. So $-(-n)$ is the number I add to $-n$ to get 0, i.e. $-(-n) = n$.
But now we're there, because

$$(-1) \times (-1) = -(-1) = 1$$

Actually: oh-oh we're half-way there.

So, what's the problem now?

I've shown that **if** I can stick on negative numbers to the positive numbers, and make the whole collection follow the same algebraic rules as the natural numbers, then $(-1) \times (-1) = 1$. But how do I know it's possible?

That's the fascinating issue of constructing the integers from the natural numbers, which this talk is, alas, too small to contain.

Maybe I should add that I'm not suggesting that teachers have this discussion (with maybe a brief introduction to abstract algebra) when they first introduce negative numbers—I'm not quite that crazy.

But I think it's a nice example of how understanding maths is a spiral process. We often benefit from coming back to the more elementary ideas with a more sophisticated point of view.

If I have a moral, it's that: revisit the things you think you understand and think about them harder. At least, I always find there's more than I previously appreciated.