Fermi problems for teaching mathematical modelling

Peter Rowlett

Sheffield Hallam University

p.rowlett@shu.ac.uk

Twitter @peterrowlett

Sunday 12th November 2017

A tweet from Alison Kiddle

Alison Kiddle
@ajk_44

Just been pondering this question: Are there more people in the world with olympics medals or with a maths degree from Cambridge University?

RETWEET 1 LIKES 3

9:15 AM - 11 Aug 2016

Fermi problems

- The story goes, Enrico Fermi had a knack for making rough estimates with very little data.
- Standard (ish) Fermi problems:
    - How many piano tuners are there in New York City?
    - How many hairs are there on a bear?
    - How many miles does a person walk in a lifetime?
    - How many people in the world are talking on their mobile phones right now?

Fermi problems for Modelling 1

- I use this (with Alison's example) as an exercise in estimation and making assumptions, comparing two unknowable things.
- Some Sheffield Hallam context Fermi problems:
  - sheets of A4 paper used by all students at Sheffield Hallam University in one semester;
  - hours taken to read every book in the Adsetts library;
  - windows on city campus;
  - PCs on city campus;
  - cats in Sheffield;
  - revenue from selling chocolate in Sheffield railway station last year in pounds;
  - cubic metres of rubbish from Sheffield Hallam that go into recycling each year.

How to approach these

- The point is to come up with a back-of-the-envelope, ‘wrong, but useful’ estimate.
- Some ‘rules’:
  - don’t look up information;
  - don’t make precise calculations using calculator or computer;
    - be imprecise – there are 300 days in a year, people are 2m tall, etc.;
    - round numbers where possible and calculate in your head.
- One approach[1] is to estimate by bounding – come up with numbers that are definitely too small and too large and then average.

[1] Fermi Estimates by Luke Muehlhauser: `http://lesswrong.com/lw/h5e/fermi_estimates/`

Averaging

- Say I think some quantity is bigger than 2 but smaller than 400.
- The arithmetic mean would be

$$\mathrm{AM}(2, 400) = \frac{2 + 400}{2} = 201.$$

- The geometric mean would be

$$\mathrm{GM}(2, 400) = \sqrt{2 \times 400} \approx 28.28.$$

- Which is a better estimate?

Averaging

- Say I think some quantity is bigger than 2 but smaller than 400.
- The arithmetic mean would be
$$\mathrm{AM}(2,400) = \frac{2+400}{2} = 201.$$
- The geometric mean would be
$$\mathrm{GM}(2,400) = \sqrt{2 \times 400} \approx 28.28.$$
- Which is a better estimate?
- AM is half the upper bound, but 100 times the lower bound. Maybe GM would be better.
- But taking the square root is not easy in your head.

Approximate Geometric Mean (AGM)

- The geometric mean of 2 and 400 is $\sqrt{2 \times 400}$.
- For the approximate geometric mean, take $2 = 2 \times 10^0$ and $400 = 4 \times 10^2$, then the AGM is

$$\frac{2+4}{2} \times 10^{\frac{2+0}{2}}$$

Approximate Geometric Mean (AGM)

- The geometric mean of 2 and 400 is $\sqrt{2 \times 400}$.
- For the approximate geometric mean, take $2 = 2 \times 10^0$ and $400 = 4 \times 10^2$, then the AGM is

$$\frac{2+4}{2} \times 10^{\frac{2+0}{2}}$$

$$= 3 \times 10^1$$

Approximate Geometric Mean (AGM)

- The geometric mean of 2 and 400 is $\sqrt{2 \times 400}$.
- For the approximate geometric mean, take $2 = 2 \times 10^0$ and $400 = 4 \times 10^2$, then the AGM is

$$\frac{2+4}{2} \times 10^{\frac{2+0}{2}}$$

$$= 3 \times 10^1$$

$$= 30 \approx 28.28 \approx \sqrt{2 \times 400}.$$

Why does this work?

- Let $A = a \times 10^x$ and $B = b \times 10^y$.
- Then

$$\mathrm{GM}(A, B) = \sqrt{AB} = \sqrt{ab \times 10^{x+y}} = \sqrt{ab} \times 10^{\frac{x+y}{2}}.$$

Why does this work?

- Let $A = a \times 10^x$ and $B = b \times 10^y$.
- Then
$$\mathrm{GM}(A, B) = \sqrt{AB} = \sqrt{ab \times 10^{x+y}} = \sqrt{ab} \times 10^{\frac{x+y}{2}}.$$
- And
$$\mathrm{AGM}(A, B) = \frac{a+b}{2} \times 10^{\frac{x+y}{2}}.$$

Why does this work?

- Let $A = a \times 10^x$ and $B = b \times 10^y$.
- Then

$$\mathrm{GM}(A, B) = \sqrt{AB} = \sqrt{ab \times 10^{x+y}} = \sqrt{ab} \times 10^{\frac{x+y}{2}}.$$

- And

$$\mathrm{AGM}(A, B) = \frac{a+b}{2} \times 10^{\frac{x+y}{2}}.$$

- Ignoring the $10^{\frac{x+y}{2}}$ term, is it obvious that, for single digit numbers > 0

$$\mathrm{GM}(a, b) = \sqrt{ab} \approx \frac{a+b}{2} = \mathrm{AM}(a, b)$$

Inequality of arithmetic and geometric means

- A standard result says:

$$\begin{aligned} 0 \leq (x-y)^2 &= x^2 - 2xy + y^2 \\ &= x^2 + 2xy + y^2 - 4xy \\ &= (x+y)^2 - 4xy \end{aligned}$$

- Hence

$$\begin{aligned} 4xy &\leq (x+y)^2 \\ \sqrt{xy} &\leq \frac{x+y}{2} \end{aligned}$$

- with equality iff $x = y$.

Brute force

- By exhaustion, it is straightforward to show the largest error is when $a = 1$ and $b = 9$. Then:

$$\sqrt{1 \times 9} = 3 \neq 5 = \frac{1+9}{2}$$

- Error then is 2.

	A	B	C	D	E	F	G	H
1	a	b	\sqrt{ab}	\frac{a+b}{2}	diff			
2	1	1	1.00	1.00	0.00		max diff	2
3	1	2	1.41	1.50	0.09			
4	1	3	1.73	2.00	0.27			
5	1	4	2.00	2.50	0.50			
6	1	5	2.24	3.00	0.76			
7	1	6	2.45	3.50	1.05			
8	1	7	2.65	4.00	1.35			
9	1	8	2.83	4.50	1.67			
10	1	9	3.00	5.00	2.00			
11	2	2	2.00	2.00	0.00			
12	2	3	2.45	2.50	0.05			
13	2	4	2.83	3.00	0.17			
14	2	5	3.16	3.50	0.34			
15	2	6	3.46	4.00	0.54			
16	2	7	3.74	4.50	0.76			
17	2	8	4.00	5.00	1.00			
18	2	9	4.24	5.50	1.26			
19	3	3	3.00	3.00	0.00			
20	3	4	3.46	3.50	0.04			
21	3	5	3.87	4.00	0.13			
22	3	6	4.24	4.50	0.26			
23	3	7	4.58	5.00	0.42			
24	3	8	4.90	5.50	0.60			
25	3	9	5.20	6.00	0.80			
26	4	4	4.00	4.00	0.00			
27	4	5	4.47	4.50	0.03			
28	4	6	4.90	5.00	0.10			
29	4	7	5.29	5.50	0.21			
30	4	8	5.66	6.00	0.34			
31	4	9	6.00	6.50	0.50			
32	5	5	5.00	5.00	0.00			
33	5	6	5.48	5.50	0.02			
34	5	7	5.92	6.00	0.08			
35	5	8	6.32	6.50	0.18			
36	5	9	6.71	7.00	0.29			
37	6	6	6.00	6.00	0.00			
38	6	7	6.48	6.50	0.02			
39	6	8	6.93	7.00	0.07			
40	6	9	7.35	7.50	0.15			
41	7	7	7.00	7.00	0.00			
42	7	8	7.48	7.50	0.02			
43	7	9	7.94	8.00	0.06			
44	8	8	8.00	8.00	0.00			
45	8	9	8.49	8.50	0.01			
46	9	9	9.00	9.00	0.00			

2 is small

- You're not likely to use this method if the numbers are of the same order of magitude, so the error is going to be at least one order of magnitude smaller than the upper bound, i.e. $10^{\frac{x+y}{2}} \ll 10^y$.
- e.g. 9 and 100:

$$\mathrm{GM}(9, 100) = \sqrt{900} = 30 \neq 50 = \mathrm{AGM}(9, 100)$$

- Which I guess makes it relatively insignificant?

Thanks for listening!

Fermi problems for teaching mathematical modelling

Peter Rowlett, Sheffield Hallam University
p.rowlett@shu.ac.uk
Twitter @peterrowlett

- Tell me:
  - Is there a nicer way to demonstrate why AGM works?
- Ask me:
  - What about the Olympic medallists and Cambridge mathematicians?
  - What do you do if the powers differ by 1?