Double Negation and the Excluded Middle

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@axiomsofchoice

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Constructive Mathematics

- "*logic as if people matter*" vs "*logic from the mind of god*"
- Fewer axioms → stronger proofs.
- Good CompSci proofs
  - computes a *witness*.

Proof (classically): There exists irrationals *a*,*b* s.t. a^b is rational

Either $\sqrt{2}^{\sqrt{2}}$ is rational or it is irrational .

If rational take $a, b = \sqrt{2}$ and we are done

If irrational take $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}$

so that $a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2$

which is also rational

<u>But which one is it?</u>

Proof (constructive): There exists irrationals *a*,*b* s.t. a^b is rational

Since $\sqrt{2}$ and $log_2(9)$ are irrational

(Proofs left as an exercise!)

We have

$$a^b = \sqrt{2}^{log_2(9)} = \sqrt{2^{log_2(9)}} = \sqrt{9} = 3$$

which is rational

Constructive Mathematics

- **¬¬A** → **A** is not sound constructively.
  - *Double negation elimination*.

- Saying
  - “*I do not dislike Marmite*”
- is *not* really the same as saying
  - “*I like Marmite*”.

Constructive Mathematics

- Asserting **A v ¬A** is unsound unless you either have a proof of A or a refutation of A
  - *Law of the excluded middle*.

- There is neither a proof nor a refutation of “N vs. NP” [as of Nov 2017]
  - http://www.claymath.org/sites/default/files/pvsnp.pdf

“Proof by Contradiction”

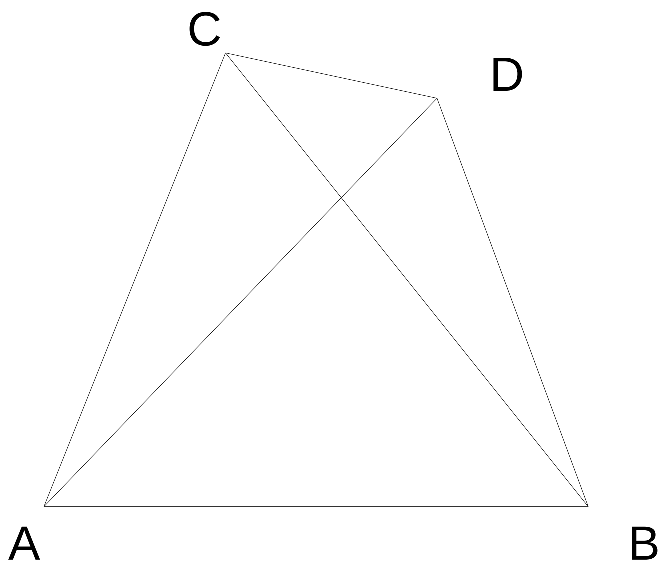
- To prove **A**, show **¬A** leads to a contradiction, hence **¬¬A** and therefore **A**.
  - Uses double negation!

Refutation by Contradiction (Proof of a negation)

- ~~To prove **A**, show **¬A** leads to a contradiction, hence **¬¬A** and therefore **A**.~~
  - ~~Uses double negation!~~
- To prove **¬A**, show **A** leads to a contradiction, hence **¬A**.
  - Perfectly reasonable!

Euclid Book I Prop 7

- Given two lines constructed on a straight line (from its extremities) and meeting in a point, there cannot be constructed on the same straight line (from its extremities) and on the same side of it, two other straight lines meeting in another point and equal to the former two respectively, namely each to that which has the same extremity with it.



Problem

- Prove **¬(¬(A v ¬A))** is true in constructively.
- “it is not the case that there is a refutation that **A** or not **A** holds”

(Due to Rob Harper, OPLSS 2012)

Further Reading

- "*Five Stages of Accepting Constructive Mathematics*" by Andrej Bauer
  - http://math.andrej.com/2016/10/10/five-stages-of-accepting-constructive-mathematics/
- Rob Harper's HoTT course at CMU, lecture 3
  - https://github.com/RobertHarper/hott-notes.git
- Rob Harper, "*Practical Foundations for Programming Languages*", chapter 30, CUP