Lyness Cycles

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(Cambridge Mathematics Education Project)

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$$1, 1, 2, 3, 5, 8, 13, \ldots$$

$$u_{n+1} = u_n + u_{n-1},\ u_1 = 1,\ u_2 = 1.$$

Order 2

$$u_1 = x,\ u_2 = y.$$

$$x,\ y,\ x + y,\ x + 2y,\ 2x + 3y,\ 3x + 5y, \ldots$$

$$\Rightarrow \quad x = 0,\ y = 0$$

$$\begin{matrix} \| & \| \\ x & y \end{matrix}$$

$$x,\ y,\ xy + 1,\ xy^2 + y + 1,\ x^2y^3 + 2xy^2 + xy + y + 2,\ldots$$

$$\text{II} \qquad\qquad \text{II}$$

$$x \qquad\qquad y$$

$x = -1,\ y = 2$

$$-1,\ 2,\ -1,\ -1,\ 2,\ -1,\ -1,\ 2\ldots$$

Locally periodic, period-3

Can we find recurrence relations that are GLOBALLY periodic?

$$x,\ y,\ \frac{y+1}{x},\ \frac{\frac{y+1}{x}+1}{y} = \frac{x+y+1}{xy},\ \frac{\frac{x+y+1}{xy}+1}{\frac{y+1}{x}} = \frac{x+1}{y},\ x,\ y,$$

$$\boxed{x,\ y,\ \frac{y+1}{x},\ \frac{x+y+1}{xy},\ \frac{x+1}{y},\ x,\ y,}$$

Globally periodic, order-2, period-5.

$$x\ ,\ \frac{2x-1}{x+1}\ ,\ \frac{x-1}{x}\ ,\ \frac{x-2}{2x-1}\ ,\ \frac{1}{1-x}\ ,\ \frac{x+1}{2-x}\ ,\ x\ ,$$

Order-1, Period-6

$$x,\ y,\ z,\ \frac{y\cdot z+1}{x\cdot y\cdot z},\ \frac{x\cdot y+y\cdot z+1}{y\cdot(y\cdot z+1)},\ \frac{(x\cdot y+1)\cdot(y\cdot z+1)}{z\cdot(x\cdot y+y\cdot z+1)},\ \frac{x\cdot y\cdot z\cdot(x\cdot y+y\cdot z+1)}{(x\cdot y+1)\cdot(y\cdot z+1)},$$

$$\frac{(x\cdot y+1)\cdot(y\cdot z+1)}{x\cdot(x\cdot y+y\cdot z+1)},\ \frac{x\cdot y+y\cdot z+1}{y\cdot(x\cdot y+1)},\ \frac{x\cdot y+1}{x\cdot y\cdot z},\ x,\ y,\ z,$$

Order-3, Period-10

Which periods are possible?

Working with real coefficients, any period is possible.

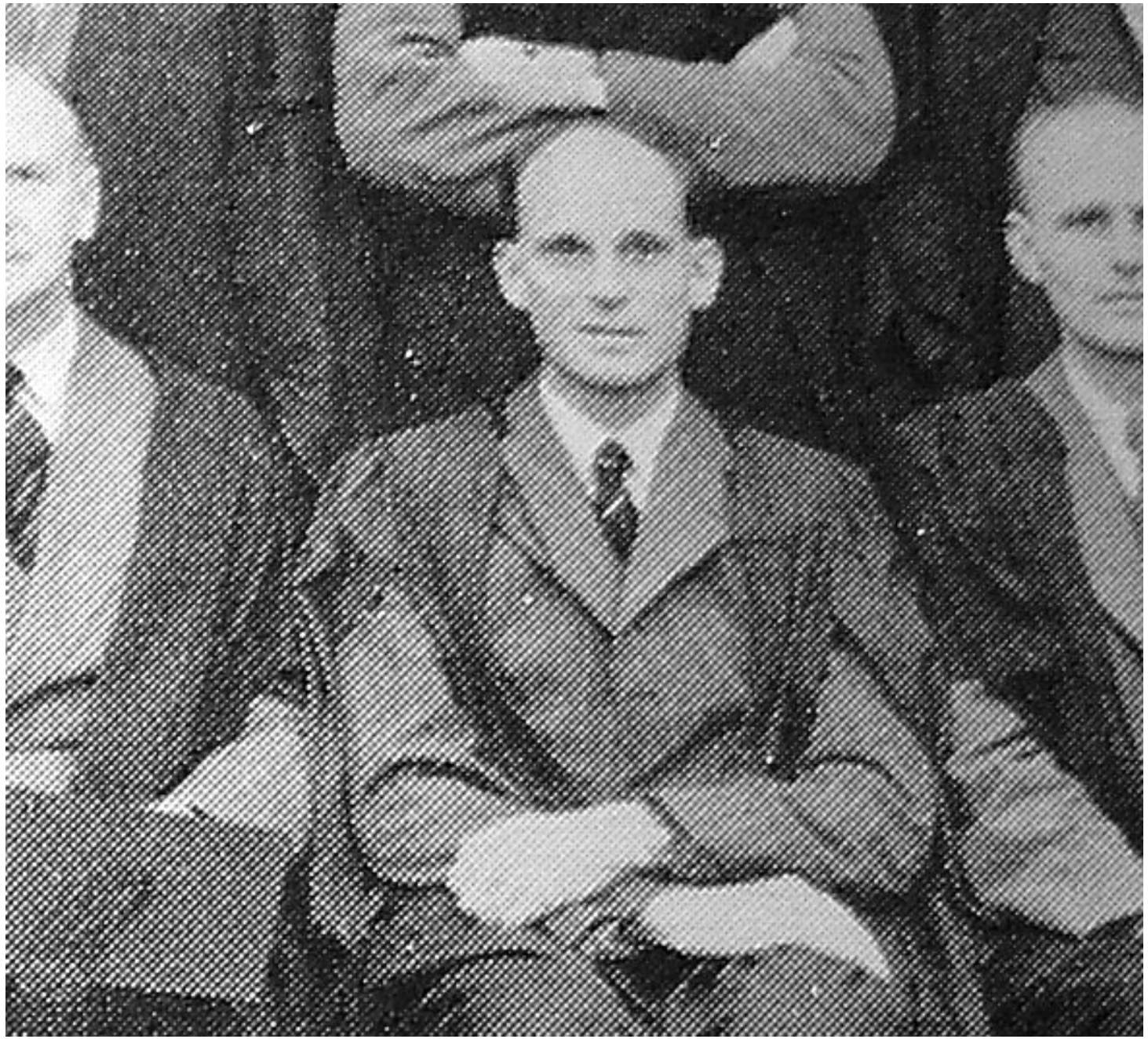
eg order-2 x, y, ky - x,...

Working with rational coefficients only?

Order 1: 1, 2, 3, 4, 6.

Order 2: 2, 3, 4, 5, 6, 8, 12.

Order 3: 3, 4, 5, 6, 8, 9, 10, 12, 18.



Robert Cranston Lyness 1909-1997.