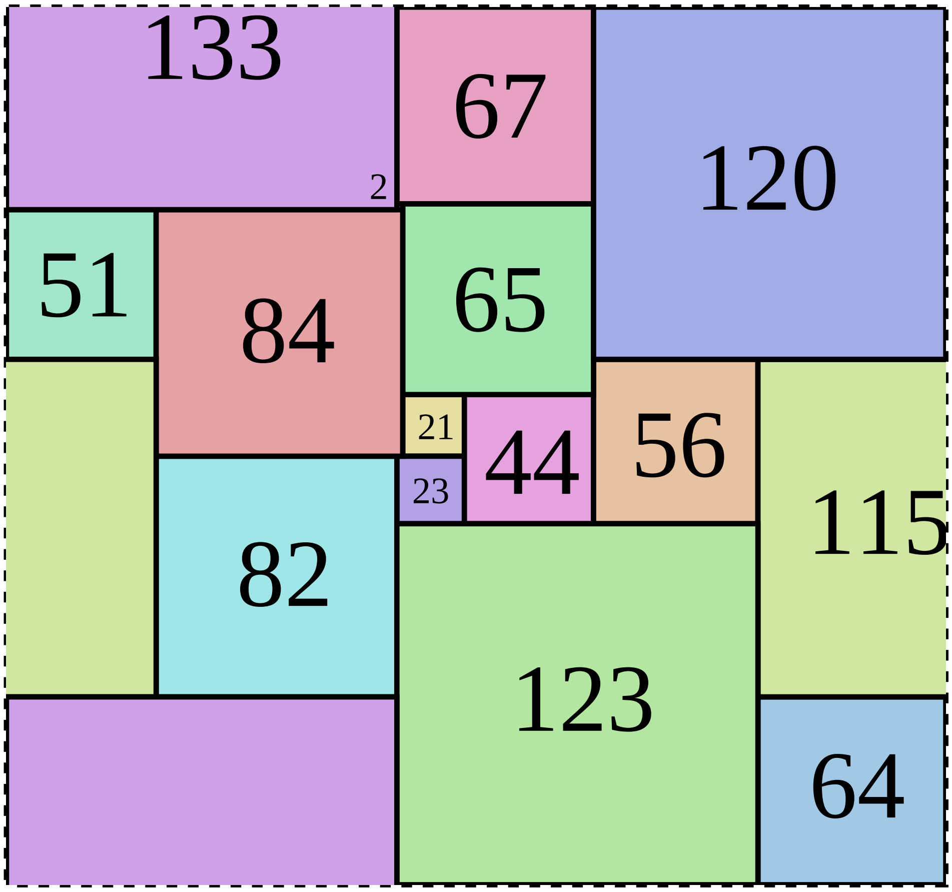
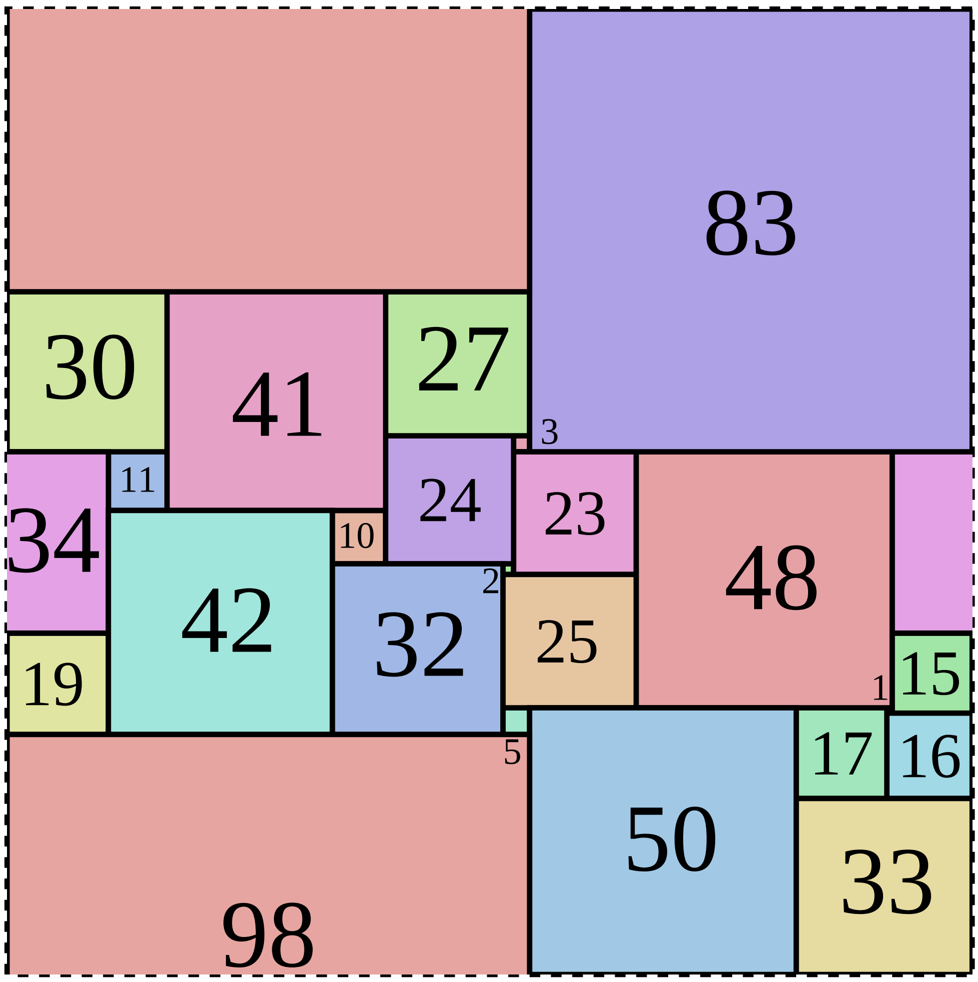
SQUARING THE TORUS

by Geoffrey H. Morley (MathsJam 2014)

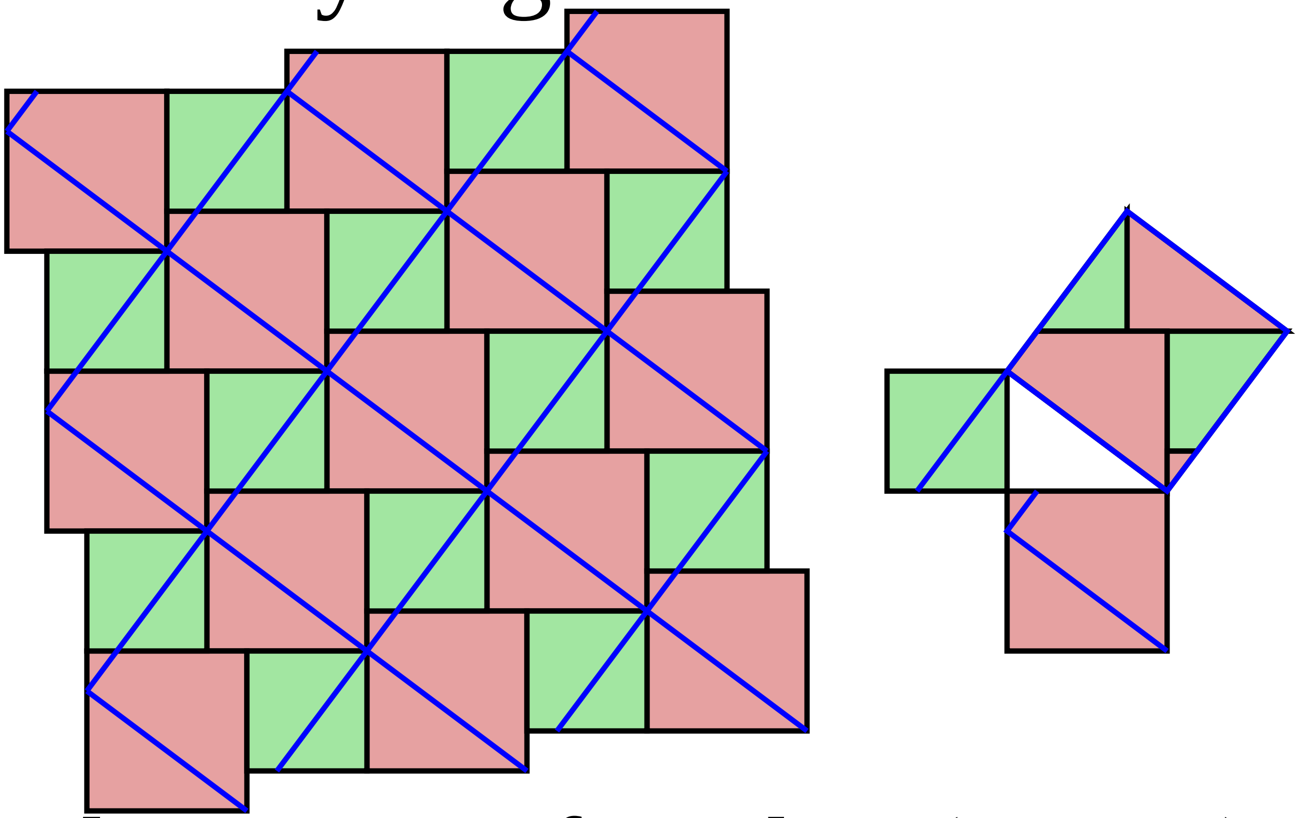
15:
320 X 299
(Gambini,
1999)



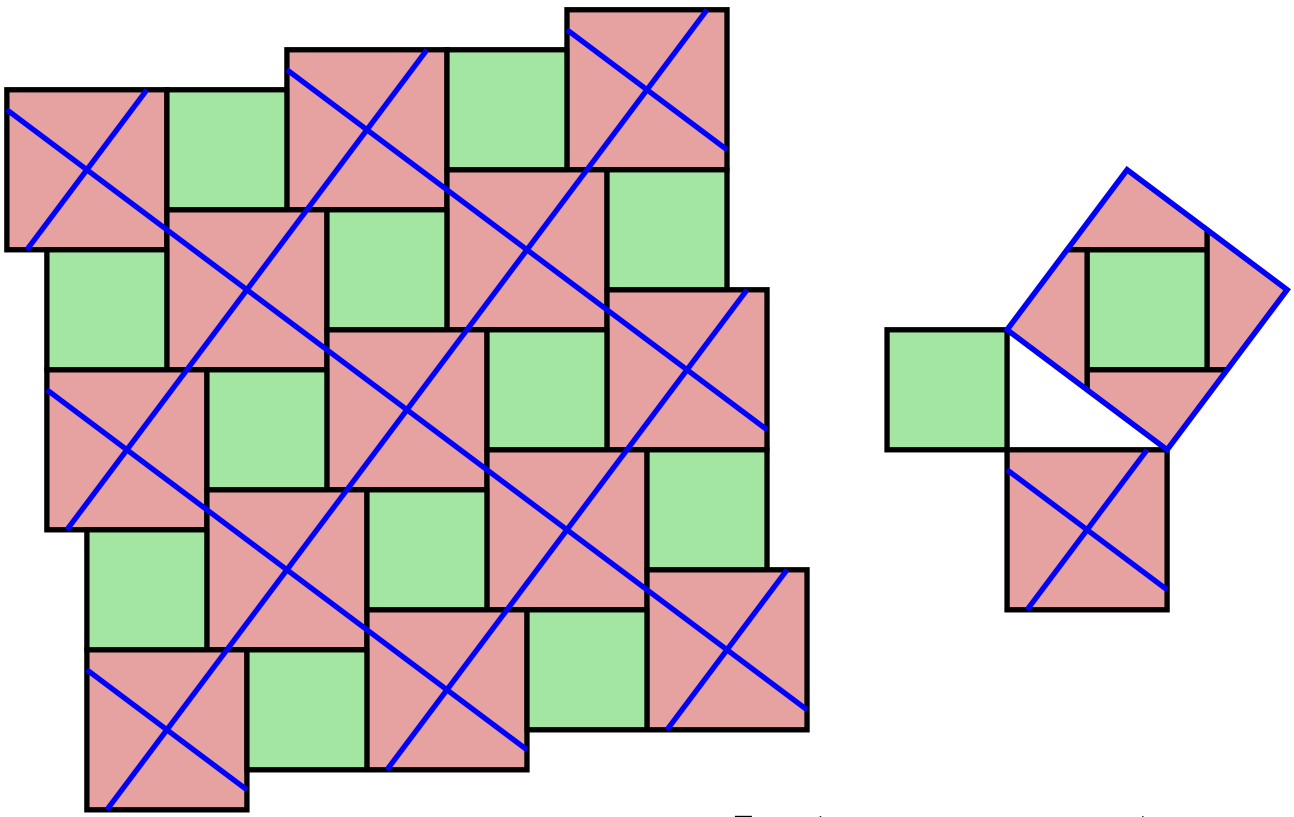
24:
181 X 181
(Gambini,
1999)



Dissection proofs of Pythagoras's theorem



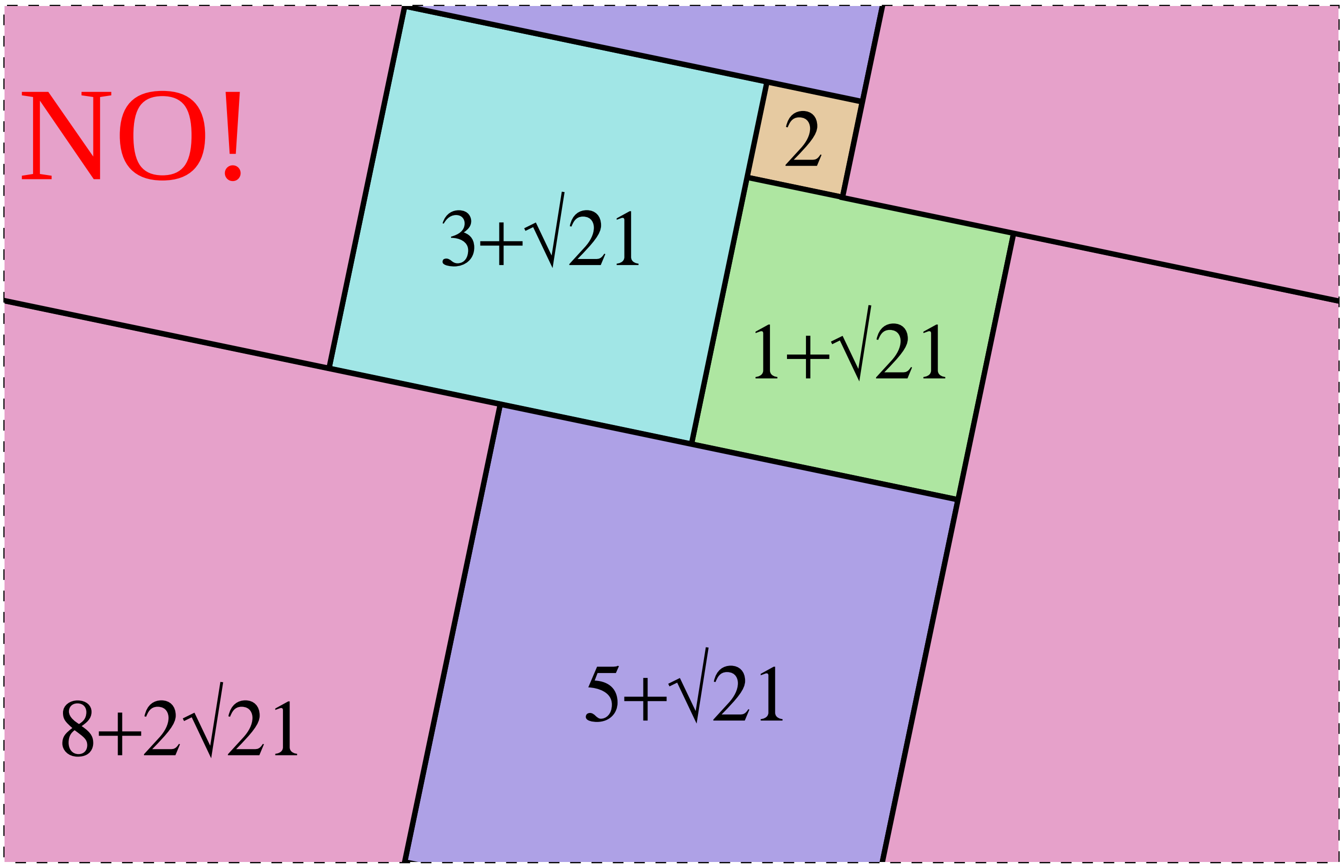
al-Nayrizi of Arabia (c. 900)



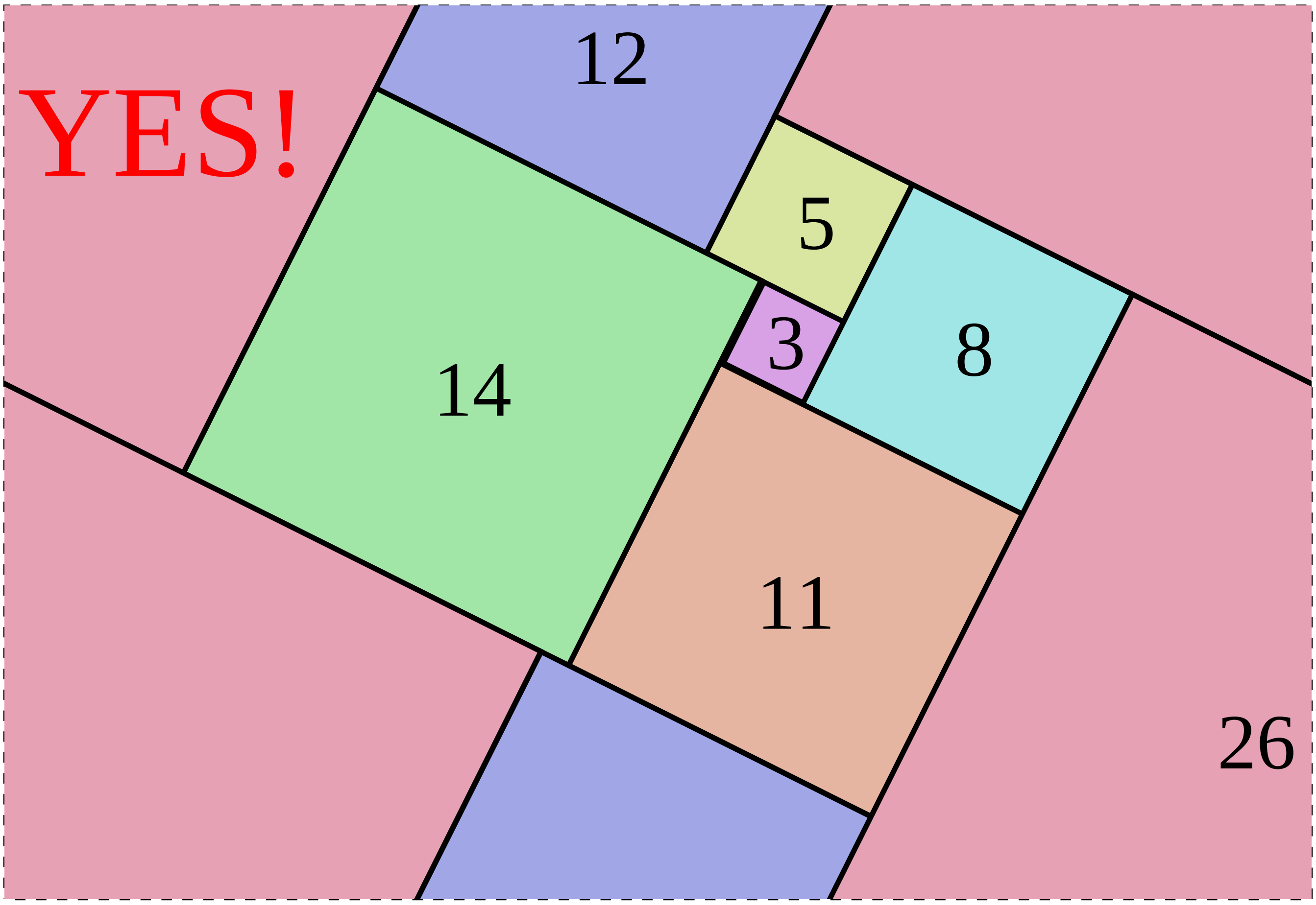
Henry Perigal (c. 1830)

Both illustrate a two-square tiling of a square torus.

TILES MUTUALLY COMMENSURABLE?

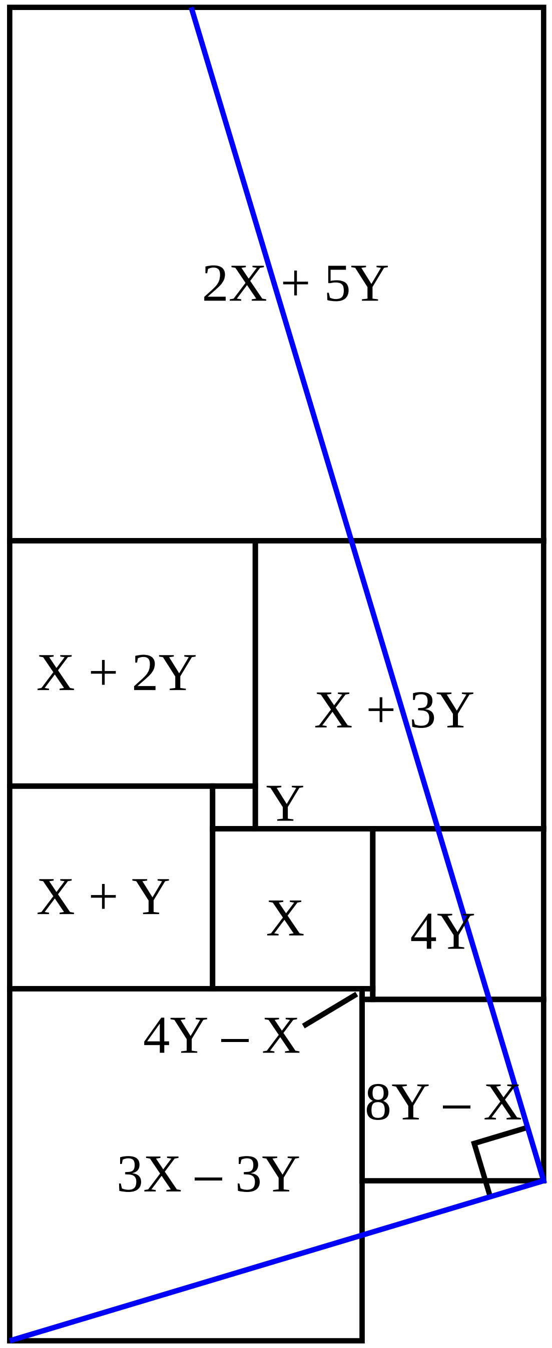


5: √(380 + 80√21) × √(170 + 30√21)

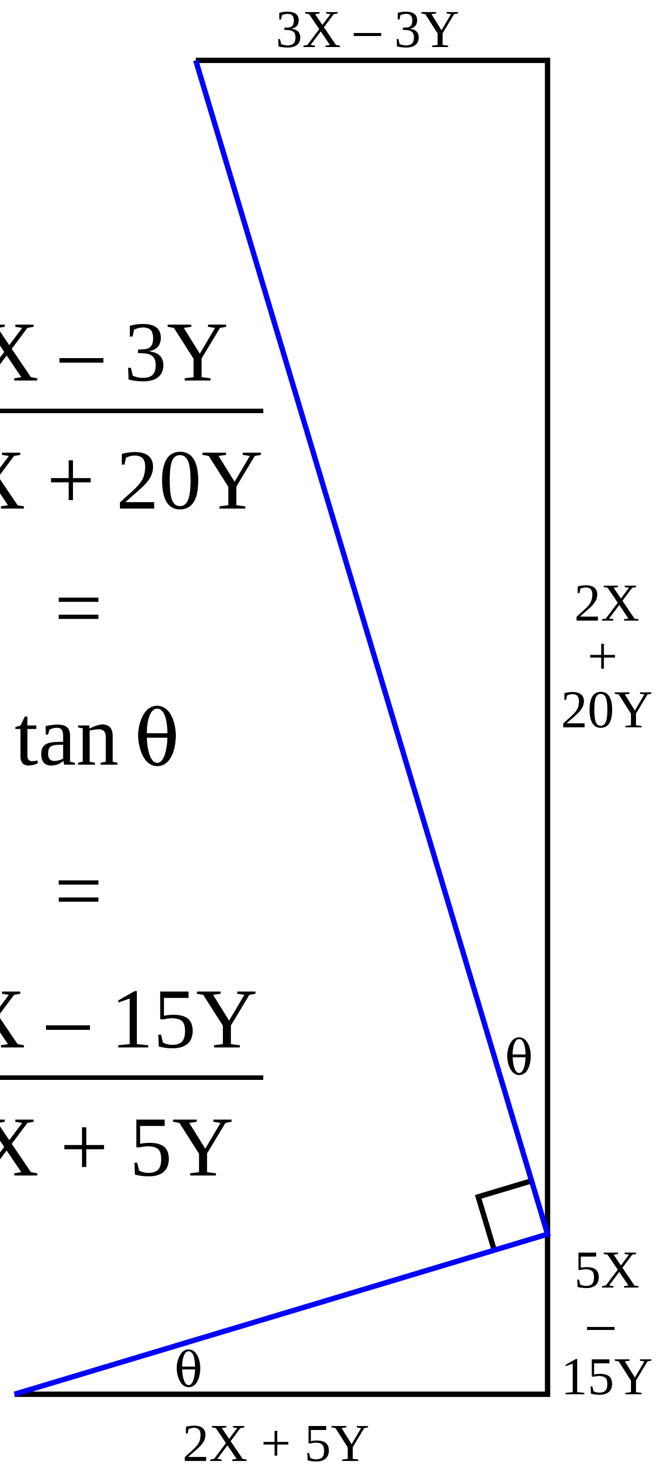


7: 19√5 × 13√5

HOW TO FIND TILE SIZES



$$\frac{3X - 3Y}{2X + 20Y} = \tan\theta = \frac{5X - 15Y}{2X + 5Y}$$

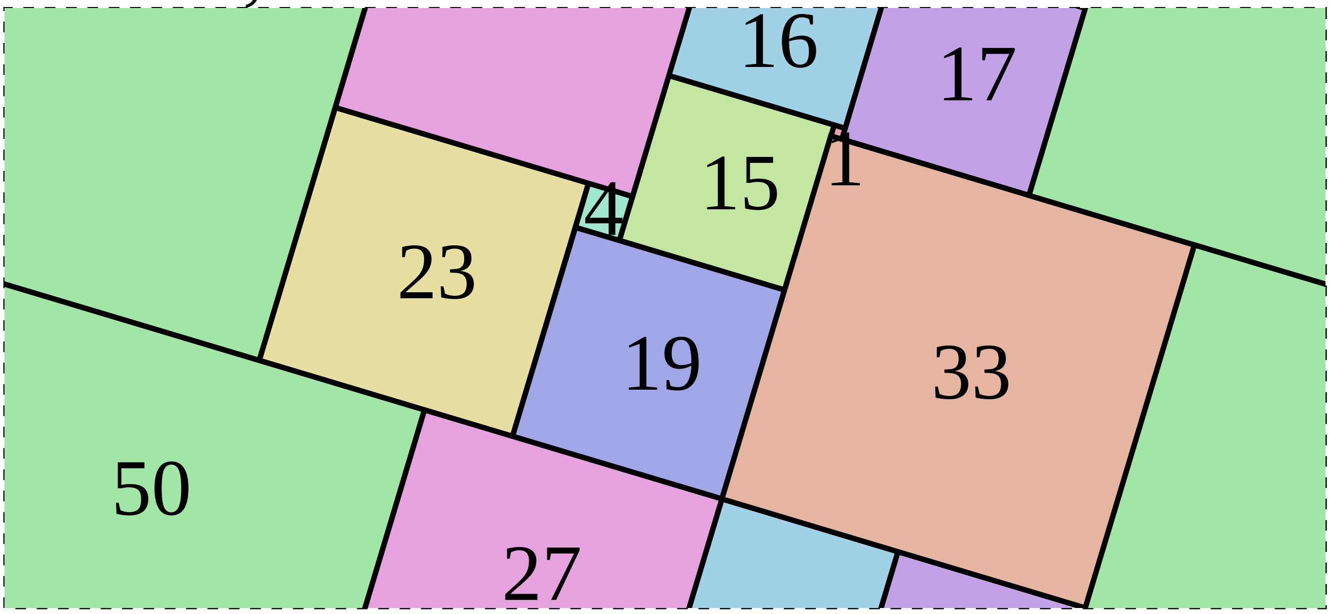


$$4\left(\frac{X}{Y}\right)^2 + 61\left(\frac{X}{Y}\right) - 285 = 0$$

$$(4X - 15Y)(X + 19Y) = 0$$

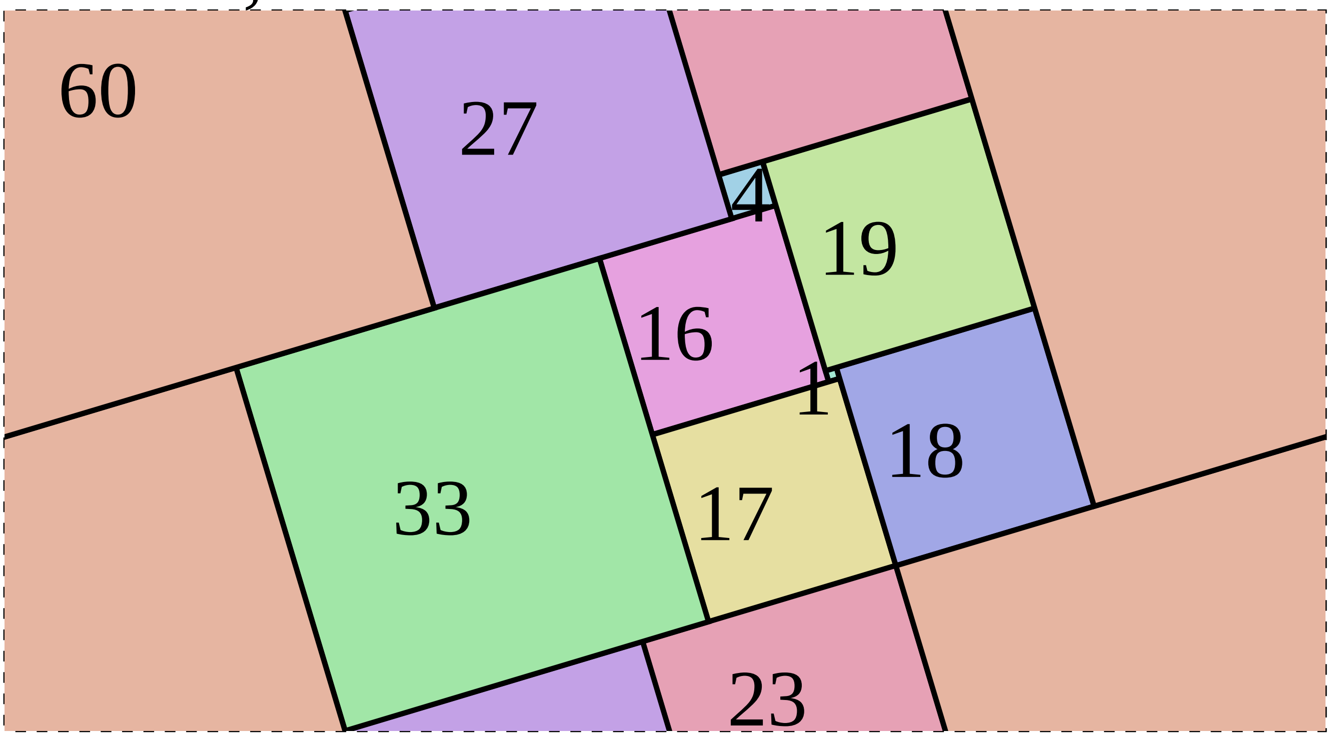
THE TWO SOLUTIONS

X = 15, Y = 4:



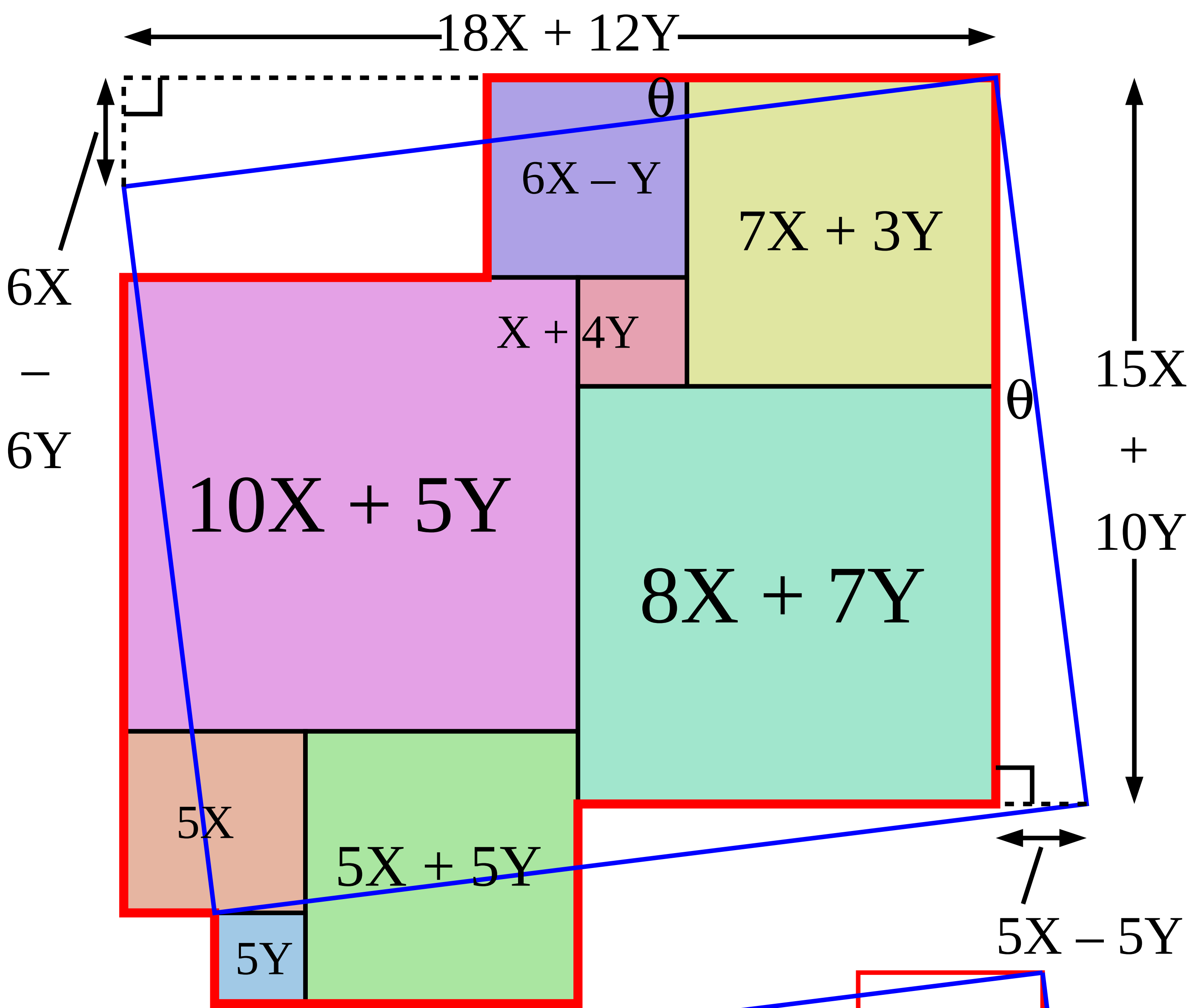
10: 11√109 × 5√109

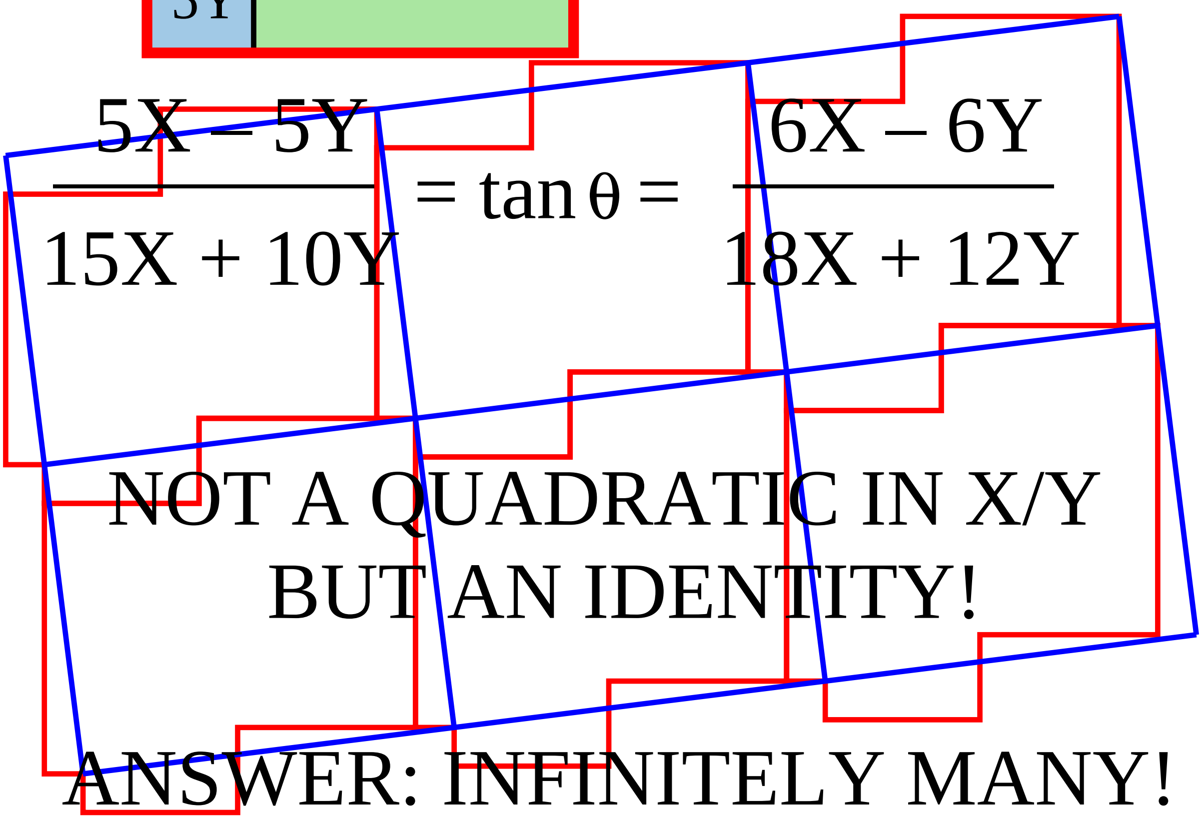
X = 19, Y = –1:



10: 11√109 × 6√109

HOW MANY SOLUTIONS FROM THIS DIAGRAM?





$$\frac{5X - 5Y}{15X + 10Y} = \tan\theta = \frac{6X - 6Y}{18X + 12Y}$$

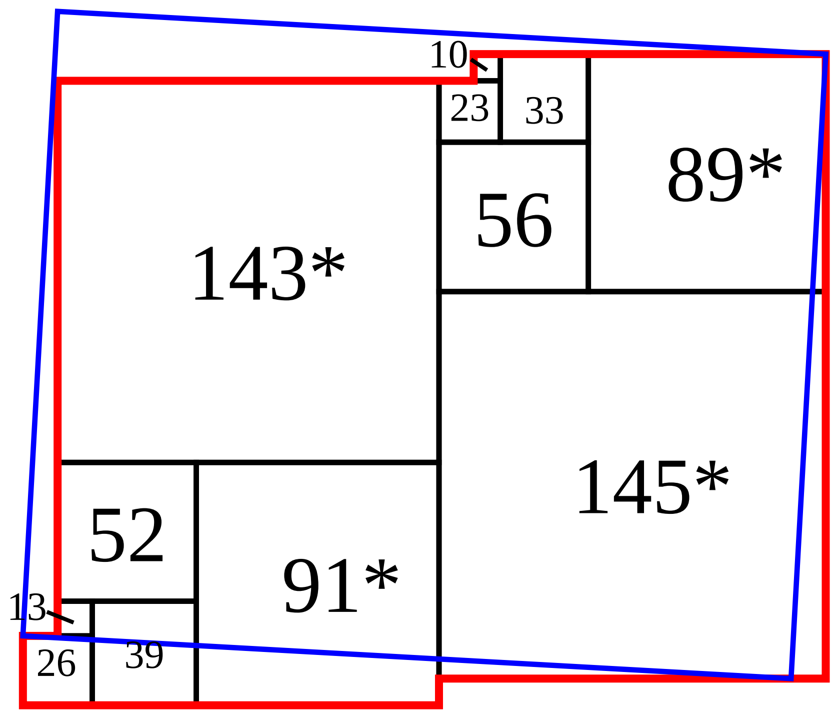
NOT A QUADRATIC IN X/Y BUT AN IDENTITY!

ANSWER: INFINITELY MANY!

MORE CONTINUOUSLY DEFORMABLE DECAGONS!

Add two squares as indicated* to each 4-square hexagon in the 8-square decagon.

This gives a 12-square continuously deformable decagon that tiles a 16 X 13 torus as in the instance below.

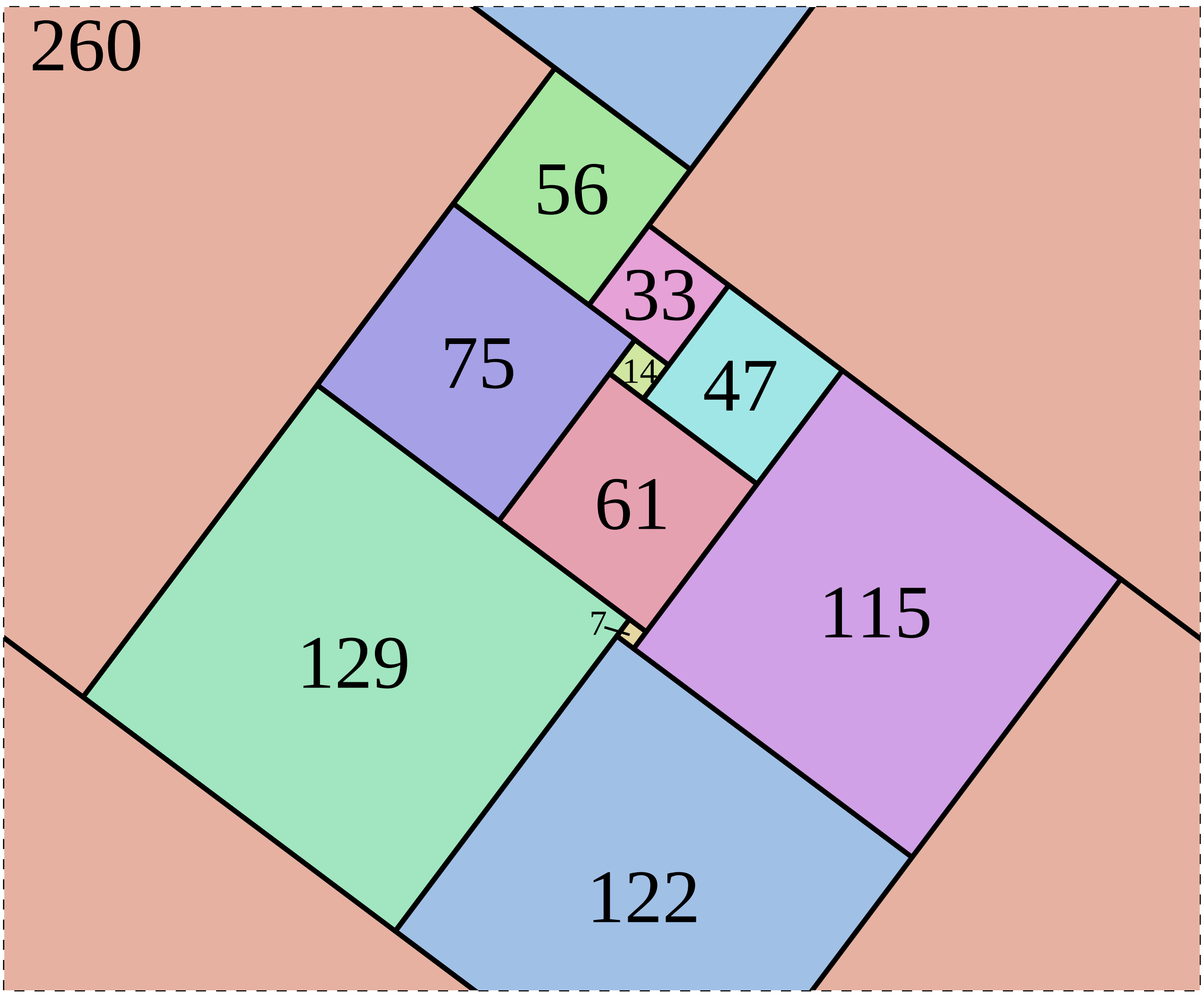


Repeat indefinitely to give a 4n-square continuously deformable decagon (n>1) that tiles a 2F(2n) X F(2n+1) torus, where F(n) is the nth positive Fibonacci number.

FIRST OF A KIND

Below is the first known squared torus which satisfies all of these conditions:

- Simple (no squared subrectangle);
- Perfect (squares unequal);
- More than two squares;
- Tile sides not parallel to rectangle sides;
- Side-lengths of the tiles are integers;
- Side-lengths of rectangle are integers.



11: 395 X 325 (Morley, 2014)

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[Apparently 1872, not 1874.]
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<THE END>